

MAT4301 – Suggested exercises from Evans – week 47

November 14, 2025

1. For a terminal-value Hamilton–Jacobi equation,

$$\begin{aligned}u_t + H(Du, x) &= 0 && \text{in } \mathbb{R}^n \times (0, T] \\u(\cdot, T) &= g && \text{on } \mathbb{R}^n \times \{0\},\end{aligned}$$

we recall that $u : \mathbb{R}^n \times [0, T] \rightarrow \mathbb{R}$ is a viscosity solution of this terminal-value problem if

- i) $u(\cdot, 0) = g$
- ii) For any $v \in C^\infty(\mathbb{R}^n \times (0, T))$, if $u - v$ has a local max (min) at $(x_0, t_0) \in \mathbb{R}^n \times (0, T)$, then

$$v_t(x_0, t_0) + H(Dv(x_0, t_0), x_0) \geq 0 \quad (\leq 0),$$

where we note that the inequalities in ii) are the reverse of the definition of viscosity solutions of initial-value HJ equations (see e.g. Definition

And for an initial-value Hamilton–Jacobi equation

$$\begin{aligned}w_t + H(Dw, x) &= 0 && \text{in } \mathbb{R}^n \times (0, T] \\w(\cdot, 0) &= g && \text{on } \mathbb{R}^n \times \{0\},\end{aligned}$$

we recall that $w : \mathbb{R}^n \times [0, T] \rightarrow \mathbb{R}$ is a viscosity solution of this initial-value problem if

- i) $w(\cdot, 0) = g$
- ii) For any $v \in C^\infty(\mathbb{R}^n \times (0, T))$, if $w - v$ has a local max (min) at $(x_0, t_0) \in \mathbb{R}^n \times (0, T)$, then

$$v_t(x_0, t_0) - H(Dv(x_0, t_0), x_0) \leq 0 \quad (\geq 0),$$

Exercise: Prove that if u is a viscosity solution of the terminal-value problem, then $w(x, t) := u(x, T - t)$ is a viscosity solution of the initial value problem $w_t - H(Dw, x) = 0$ (note the $-$ sign), with initial condition $w(x, 0) = g(x)$.

Hint: Pair the test function $v(x, t)$ with $\tilde{v}(x, t) := v(x, T - t)$ when connecting condition ii) for appropriate terminal-value and initial-value problems.

2. Consider a given optimal control problem with cost functional

$$C_{x,t}[\alpha] = \int_t^T r(X(s), \alpha(s)) ds + g(X(T))$$

with control-dependent dynamics

$$\begin{aligned}\dot{X}(s) &= f(X(s), \alpha(s)) \quad s > t \\ X(t) &= x,\end{aligned}$$

set of admissible controls $\mathcal{A} = \{\alpha : [0, T] \rightarrow A \subset \mathbb{R}^m \mid \alpha \text{ is measurable}\}$ with $A \subset \mathbb{R}^m$ a compact set, and smooth and bounded functions f, r and g . Suppose that we have access to the value function $u(x, t) = \inf_{\alpha \in \mathcal{A}} C_{x,t}[\alpha]$ and that u is smooth. Prove that if we for each $s > t$ define/pick $\alpha^*(s) \in A$ s.t.

$$Du(X^*(s), s) \cdot f(X^*(s), \alpha^*(s)) + r(X^*(s), \alpha^*(s)) = \min_{a \in A} Du(X^*(s), s) \cdot f(X^*(s), a) + r(X^*(s), a),$$

where X^* denotes the dynamics governed by the control α^* , then it formally holds that $u(x, t) = C_{x,t}[\alpha^*]$.

Hint: Consider the evolution of $u(X^*(s), s)$.

3. Evans problem 10.4.5.

Hint: a) u is differentiable everywhere except at $x = 0$ and u' has left and right limits for all x . Show also that $u - v$ cannot have a local min at $x_0 = 0$.

c) Find a Hamiltonian, and construct a viscosity constraint that is consistent with the one in a) (for its Hamiltonian).

4. Evans problem 10.4.6. Recall that $\text{dist}(x, \partial U) := \inf_{q \in \partial U} |x - q|$

Hint: As ∂U is a closed set, we have for any $x \in U$, that there exists a $z \in \partial U$ s.t. $\text{dist}(x, \partial U) = |x - z|$.

5. Evans problem 10.4.10 part (a).

6. (Challenging) Evans problem 10.4.11. (The λu appearing in “condition ii)” of the viscosity solution may look like a typo, but is correct.)

You may use the dynamic programming property (that for this time-independent HJB equation) that for any $h > 0$,

$$u(x) = \inf_{\alpha} \left[\int_0^h r(X(s), \alpha(s)) ds + e^{-\lambda h} u(X(h)) \right].$$