

MAT4301 – Suggested exercises from Evans – week 47

November 22, 2025

1. For a terminal-value Hamilton–Jacobi equation,

$$\begin{aligned} u_t + H(Du, x) &= 0 & \text{in } \mathbb{R}^n \times (0, T] \\ u(\cdot, T) &= g & \text{on } \mathbb{R}^n \times \{0\}, \end{aligned}$$

we recall that $u : \mathbb{R}^n \times [0, T] \rightarrow \mathbb{R}$ is a viscosity solution of this terminal-value problem if

- i) $u(\cdot, 0) = g$
- ii) For any $v \in C^\infty(\mathbb{R}^n \times (0, T))$, if $u - v$ has a local max (min) at $(x_0, t_0) \in \mathbb{R}^n \times (0, T)$, then

$$v_t(x_0, t_0) + H(Dv(x_0, t_0), x_0) \geq 0 \quad (\leq 0),$$

where we note that the inequalities in ii) are the reverse of the definition of viscosity solutions of initial-value HJ equations (see e.g. Definition

And for an initial-value Hamilton–Jacobi equation

$$\begin{aligned} w_t + H(Dw, x) &= 0 & \text{in } \mathbb{R}^n \times (0, T] \\ w(\cdot, 0) &= g & \text{on } \mathbb{R}^n \times \{0\}, \end{aligned}$$

we recall that $w : \mathbb{R}^n \times [0, T] \rightarrow \mathbb{R}$ is a viscosity solution of this initial-value problem if

- i) $w(\cdot, 0) = g$
- ii) For any $v \in C^\infty(\mathbb{R}^n \times (0, T))$, if $w - v$ has a local max (min) at $(x_0, t_0) \in \mathbb{R}^n \times (0, T)$, then

$$v_t(x_0, t_0) - H(Dv(x_0, t_0), x_0) \leq 0 \quad (\geq 0),$$

Exercise: Prove that if u is a viscosity solution of the terminal-value problem, then $w(x, t) := u(x, T - t)$ is a viscosity solution of the initial value problem $w_t - H(Dw, x) = 0$ (note the $-$ sign), with initial condition $w(x, 0) = g(x)$.

Hint: Pair the test function $v(x, t)$ with $\tilde{v}(x, t) := v(x, T - t)$ when connecting condition ii) for appropriate terminal-value and initial-value problems.

2. Consider a given optimal control problem with cost functional

$$C_{x,t}[\alpha] = \int_t^T r(X(s), \alpha(s)) ds + g(X(T))$$

with control-dependent dynamics

$$\begin{aligned}\dot{X}(s) &= f(X(s), \alpha(s)) \quad s > t \\ X(t) &= x,\end{aligned}$$

set of admissible controls $\mathcal{A} = \{\alpha : [0, T] \rightarrow A \subset \mathbb{R}^m \mid \alpha \text{ is measurable}\}$ with $A \subset \mathbb{R}^m$ a compact set, and smooth and bounded functions f, r and g . Suppose that we have access to the value function $u(x, t) = \inf_{\alpha \in \mathcal{A}} C_{x,t}[\alpha]$ and that u is smooth. Prove that if we for each $s > t$ define/pick $\alpha^*(s) \in A$ s.t.

$$Du(X^*(s), s) \cdot f(X^*(s), \alpha^*(s)) + r(X^*(s), \alpha^*(s)) = \min_{a \in A} Du(X^*(s), s) \cdot f(X^*(s), a) + r(X^*(s), a),$$

where X^* denotes the dynamics governed by the control α^* , then it formally holds that $u(x, t) = C_{x,t}[\alpha^*]$.

Hint: Consider the evolution of $u(X^*(s), s)$.

3. Evans problem 10.4.5.

Hint: a) u is differentiable everywhere except at $x = 0$ and u' has left and right limits for all x . Show also that $u - v$ cannot have a local min at $x_0 = 0$.

c) Find a Hamiltonian, and construct a viscosity constraint that is consistent with the one in a) (for its Hamiltonian).

4. Evans problem 10.4.6. Recall that $\text{dist}(x, \partial U) := \inf_{q \in \partial U} |x - q|$

Hint: As ∂U is a closed set, we have for any $x \in U$, that there exists a $z \in \partial U$ s.t. $\text{dist}(x, \partial U) = |x - z|$.

5. Evans problem 10.4.10 part (a).

6. (Challenging) Evans problem 10.4.11. (The λu appearing in “condition ii)” of the viscosity solution may look like a typo, but is correct.)

You may use the dynamic programming property (that for this time-independent HJB equation) that for any $h > 0$,

$$u(x) = \inf_{\alpha} \left[\int_0^h r(X(s), \alpha(s)) ds + e^{-\lambda h} u(X(h)) \right].$$

Solution:

1. Condition i) is obviously satisfied for w , as $w(x, 0) = u(x, T - 0) = g(x)$. For condition ii), note that if $v \in C^\infty(\mathbb{R}^n \times (0, T))$ and $w - v$ has a local max at $(x_0, t_0) \in \mathbb{R}^n \times (0, T)$, then $u(x, t) - \tilde{v}(x, t)$ with $\tilde{v}(x, t) := v(x, T - t)$ has a local max at $(x_0, T - t_0)$. By condition ii) for the terminal-value problem,

$$\tilde{v}_t(x_0, T - t_0) + H(D\tilde{v}(x_0, T - t_0), x_0) \geq 0.$$

Noting that $D\tilde{v}(x_0, T - t_0) = Dv(x_0, t_0)$ and $\tilde{v}_t(x_0, T - t_0) = -v_t(x_0, t)$,

$$-v_t(x_0, t_0) + H(Dv(x_0, t_0), x_0) \geq 0 \implies v_t(x_0, t_0) - H(Dv(x_0, t_0), x_0) \leq 0,$$

verifying that condition ii) holds for local maxima for w . The local min condition is proved similarly.

2. Recall that

$$\begin{aligned}\dot{X}^*(s) &= f(X^*(s), \alpha^*(s)) \quad s > t \\ X^*(t) &= x,\end{aligned}$$

and that u solves the Hamilton–Jacobi–Bellman equation

$$\begin{aligned}u_t + \min_{a \in A} \min_{a \in A} Du(x, s) \cdot f(x, a) + r(x, a) &= 0 \quad (x, t) \in \mathbb{R}^n \times [0, T) \\ u(x, T) &= g(x) \quad x \in \mathbb{R}^n.\end{aligned}$$

Using these properties, we formally have that

$$\frac{d}{ds} u(X^*(s), s) = Du(X^*(s), s) \cdot \dot{X}^*(s) + u_t(X^*(s), s) = -r(X^*(s), \alpha^*(s)).$$

Hence,

$$u(X^*(T), T) - u(x, t) = - \int_t^T r(X^*(s), \alpha^*(s)) ds,$$

which yields

$$u(x, t) = \int_t^T r(X^*(s), \alpha^*(s)) ds + g(X^*(T)) = C_{x,t}[\alpha^*].$$

3. If $u - v$ has a local max at $x_0 \in (-1, 1)$, then $u'(x_0-) - v'(x_0+) := \lim_{\delta \downarrow 0} u'(x_0 - \delta) - v'(x_0 - \delta) \geq 0$ and for the right limit, $u'(x_0+) - v'(x_0+) \leq 0$. Further, since the right and left limits of $u'(x \pm) = -\text{sign}(x \pm)$ is well defined for all x and $v'(x \pm) = v'(x)$ by continuity, we obtain that

$$-\text{sign}(x_0+) \leq v'(x_0) \leq -\text{sign}(x_0-) \implies |v'(x_0)| \leq 1.$$

This shows that $u(x) = 1 - |x|$ satisfies the viscosity solution criterion for local maxima.

If x_0 is a local min, then

$$u'(x_0-) - v'(x_0) \leq 0 \implies v'(x_0) \geq -\text{sign}(x_0-),$$

and

$$u'(x_0+) - v'(x_0) \geq 0 \implies v'(x_0) \leq -\text{sign}(x_0+).$$

For $x_0 \neq 0$, this becomes $v'(x_0) = -\text{sign}(x_0)$, consistent with $|v'(x_0)| \geq 1$. For $x_0 = 0$, it becomes $1 \geq v'(0) \leq -1$, which means $u - v$ cannot have a local min at that point, since there exists no function v

satisfying those the lower and upper bounds simultaneously. We conclude that for all possible local min of $u - v$, it holds that $|v'(x_0)| \geq 1$.

For the smooth function $v(x) = -x^2$, we see that $\tilde{u}(x) - v(x) = |x| - 1 + x^2$, has a strict minimum at $x_0 = 0$, where $v'(x_0) = 0$. This does not satisfy the local-min viscosity constraint $|v'(x_0)| \geq 1$.

- (b) Noting that $\tilde{H}(\tilde{u}') = -|\tilde{u}'| + 1$, a uniformly continuous and bounded u is a viscosity solution of $(**)$ if it satisfies the BC and for each smooth v , if $u - v$ has a local max (min) at x_0 , then

$$\tilde{H}(v'(x_0)) \leq 0 \quad (\geq 0)$$

If x_0 is a local max of $\tilde{u} - \tilde{v}$, then x_0 also is a local min of $-(\tilde{u} - \tilde{v}) = u - v$ where $v := -\tilde{v}$. Since u is a viscosity solution and $\tilde{v}$ smooth, $|\tilde{v}'(x_0)| \geq 1$. We conclude that $|v'(x_0)| \geq 1$ or equivalently $\tilde{H}(v'(x_0)) \leq 0$. The local-min condition is verified similarly.

- (c) The problems have different viscosity solutions because their Hamiltonians have opposite signs: a) $H(u') = |u'| - 1$ versus c) $\tilde{H}(u') = -|u'| + 1$. Viscosity solutions are defined in terms of inequality conditions, and one cannot simply multiply one Hamiltonian with -1 to obtain problem-equivalence with the other Hamiltonian, as such a multiplication would reverse the inequality conditions (if there were equality-conditions instead, then the problems would be equivalent).

4. Take an arbitrary pair of points $x, y \in U$ such that $\text{dist}(x, \partial U) \geq \text{dist}(y, \partial U)$, and note that

$$\begin{aligned} \text{dist}(x, \partial U) &= \inf_{z \in \partial U} \text{dist}(x, z) \\ &\leq \inf_{z \in \partial U} \text{dist}(x, y) + \text{dist}(y, z) \leq \text{dist}(x, y) + \text{dist}(y, \partial U). \end{aligned}$$

Consequently,

$$|u(x) - u(y)| = \text{dist}(x, \partial U) - \text{dist}(y, \partial U) \leq \text{dist}(x, y) = |x - y|$$

so u is Lipschitz continuous. If $x_0 \in U$ is local max of $u - v$, then for any vector $\eta \in \mathbb{R}^n$ with $|\eta| = 1$ and $h > 0$, $u(x_0 + h\eta) - v(x_0 + h\eta) \leq u(x_0) - v(x_0)$. This yields

$$-\frac{v(x_0 + h\eta) - v(x_0)}{h} \leq \frac{u(x_0 + h\eta) - u(x_0)}{h} \leq 1.$$

Taking the limit $h \downarrow 0$ yields

$$-Dv(x_0) \cdot \eta \leq 1.$$

Take $\eta = -Dv(x_0)$ to conclude that $|Dv(x_0)| \leq 1$.

If x_0 is a local min of $u - v$, then we have similarly as above that for all small $h > 0$ and unit vectors $\eta \in \mathbb{R}^n$ that the following holds

$$\frac{u(x_0 + h\eta) - u(x_0)}{h} \geq \frac{v(x_0 + h\eta) - v(x_0)}{h}. \quad (1)$$

Since ∂U is a closed set, there exists a $z \in \partial U$ s.t.

$$\text{dist}(x, \partial U) = \text{dist}(x, z).$$

For the vector $\eta = (z - x)/|z - x|$, it therefore holds for any $h \in (0, |z - x|)$ that $u(x) = h + u(x + h\eta)$ (as $(x + h\eta)_{h \in [0, |z - x|]}$ describes the geodesic curve from x to ∂U). Applying this in (1) and taking the limit $h \downarrow 0$ yields

$$1 \geq Dv(x_0) \cdot \eta \leq |Dv(x_0)|$$

where Cauchy-Schwarz was applied in the last inequality.

5. By the uniform bound $|r(x, a)| \leq C$,

$$|u(x)| = |\inf_{\alpha} C_x[\alpha]| \leq \sup_{\alpha} |C_x[\alpha]| \leq \sup_{\alpha} \int_0^{\infty} e^{-\lambda s} |r(X(s), \alpha(s))| ds \leq C/\lambda.$$

Let $C_f > 0$ denote the x-Lipschitz constant for f and $C_r > 0$ the x-Lipschitz constant for r and take two arbitrary points $x, y \in \mathbb{R}^n$. Let $\varepsilon > 0$ and let $\alpha \in$ be such that

$$C_x[\alpha] \leq u(x) + \varepsilon.$$

Let further X denote the dynamics $\dot{X}(s) = f(X(s), \alpha(s))$ for $s > 0$ with $X(0) = x$, and let $\hat{X}$ denote the dynamics $\dot{\hat{X}}(s) = f(\hat{X}(s), \alpha(s))$ for $s > 0$ with $\hat{X}(0) = y$. Then

$$\begin{aligned} u(y) - u(x) &\leq C_y[\alpha] - C_x[\alpha] + \varepsilon \\ &\leq \int_0^{\infty} e^{-\lambda s} |r(\hat{X}(s), \alpha(s)) - r(X(s), \alpha(s))| ds + \varepsilon \\ &\leq C_r \int_0^{\infty} e^{-\lambda s} |\hat{X}(s) - X(s)| ds + \varepsilon \end{aligned}$$

As in the proof of Lemma 10.3.1, one can show using Cauchy-Schwarz that

$$\frac{d}{ds} |\hat{X} - X|^2 \leq 2C_f |\hat{X} - X|^2 \implies |\hat{X}(s) - X(s)| \leq e^{C_f s} |y - x|.$$

Provided $C_f < \lambda$, this yields

$$u(y) - u(x) \leq C_r |y - x| \int_0^{\infty} e^{(C_f - \lambda)s} ds + \varepsilon \leq \underbrace{\frac{C_r}{\lambda - C_f}}_{=: C} |y - x| + \varepsilon.$$

Reversing the roles of y and x , we obtain

$$|u(y) - u(x)| \leq C|y - x| + \varepsilon,$$

and since argument applies for any $\varepsilon > 0$, u is Lipschitz continuous.

6. Suppose x is a local max of $u - v$. Then there exists an $a \in A$ and $\delta, \theta > 0$ s.t.

$$\lambda u(y) - f(y, a) \cdot Dv(y) - r(y, a) \geq \theta \quad (2)$$

and $u(y) - v(y) \leq u(x) - v(x)$ for all $y \in B(x, \delta)$.

Let $\alpha(s) = a$ be a constant control with the dynamics $\dot{X}(s) = f(X(s), \alpha(s)) = f(X(s), a)$ for $s > 0$ and $X(0) = x$ and let $h \in (0, \delta)$ be so small that $|X(s) - x| \leq \delta$ for all $s \leq \delta$. Then

$$\lambda u(X(s)) - Dv(X(s)) \cdot f(X(s), a) + r(X(s), a) \geq \theta \quad \forall s \in [0, \delta],$$

and

$$u(X(h)) - u(x) \leq v(X(h)) - v(x) = \int_0^h Dv(X(s)) \cdot f(X(s), a) ds$$

By dynamic programming, we further know that

$$u(x) \leq \int_0^h r(X(s), a) ds + e^{-\lambda h} u(X(h)) = \int_0^h r(X(s), a) ds + u(X(h)) - u(x) + (e^{-\lambda h} - 1)u(X(h)).$$

This yields,

$$\begin{aligned} 0 &\leq \int_0^h r(X(s), a) ds + (e^{-\lambda h} - 1)u(X(h)) + u(X(h)) - u(x) \\ &\leq \int_0^h r(X(s), a) + Dv(X(s)) \cdot f(X(s), a) ds + (e^{-\lambda h} - 1)u(X(h)) \end{aligned}$$

and taking the limit $h \downarrow 0$ leads to the contradiction

$$0 \leq r(x, a) + Dv(x) \cdot f(x, a) - \lambda u(x) \stackrel{(2)}{\leq} -\theta.$$

Hence (2) cannot hold for any θ strictly positive and the local-max condition for viscosity solutions holds for u . The local-min condition is verified similarly.