

MAT4301: Mandatory assignment, autumn 2026

Existence of a solution of the Dirichlet problem in (more) general domains.

We consider the Dirichlet problem

$$(1) \quad \begin{cases} \Delta u = 0, & x \in U \\ u = g & x \in \partial U, \end{cases}$$

where U is a domain (open, bounded and connected) in $\mathbb{R}^n$ and g is a continuous function. The exercise is a proof of existence of a solution for fairly general domains U . It consists of many steps, you must understand the black text and answer the questions in [blue](#).

The deadline for the assignment is Thursday, November 5th. Use *Canvas* to submit your answer.

Exercise 1. Let $\{u_k\}_{k=1}^\infty$ be a sequence of bounded harmonic functions in $C^2(\bar{U})$, where $U \subset \mathbb{R}^n$ is a domain. For each $z \in U$, show that there is an $r > 0$ such that $\{u_k\}$ contains a subsequence which converges uniformly in $B(z, r)$. **Hint:** Use the Arzelà-Ascoli theorem.

Exercise 2. Let $\{u_k\}_{k=1}^\infty$ be the uniformly convergent subsequence, define $u(x) = \lim_k u_k(x)$ for $x \in B(z, r)$. Show that $u \in C^2(B(z, r))$ and that u is harmonic in $B(z, r)$.

In the remaining part of the mandatory exercise we shall show that there exists a solution to the Dirichlet problem (1) for quite general domains U . For the special domain $B(z, r)$ we have a formula for the solution

$$(2) \quad u(x) = \frac{r^2 - |x - z|^2}{n\alpha(n)r} \int_{\partial B(z, r)} \frac{g(y)}{|x - z - y|^n} dS(y).$$

You do not have to show this formula (it is done in Evans), nor do you have to use it. It is enough to know that a solution *exists*.

We call a function *subharmonic* at $x \in U$ if there exists an $r > 0$ such that for all $s \leq r$ we have

$$(3) \quad u(x) \leq \int_{\partial B(x, s)} u(y) dS(y).$$

If u is subharmonic for all $x \in U$ we say that u is subharmonic in U .

Exercise 3. Show that if u is subharmonic in U then

(a)

$$\max_{x \in \bar{U}} u(x) = \max_{x \in \partial U} u(x).$$

(b) Furthermore, if there exists a point $x_0 \in U$ such that $u(x_0) = \max_{x \in \bar{U}} u(x)$, then u is constant in U .

Define the set

$$S_g = \left\{ u \in C(\bar{U}) \mid u \text{ is subharmonic in } U \text{ and } u \leq g \text{ on } \partial U \right\},$$

and the function

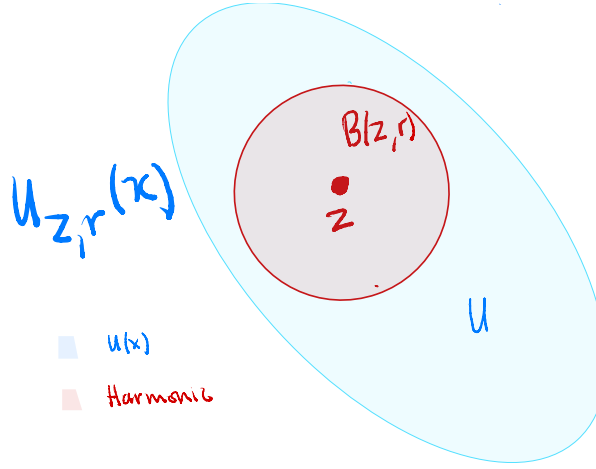
$$w_g(x) = \sup \left\{ u(x) \mid u \in S_g \right\}.$$

Exercise 4. Show that $w_g(x)$ exists for all $x \in U$, and that $w_g(x) < \infty$ for all $x \in U$.

Now for $u \in C(U)$ and z, r such that $B_0(z, r) \subset U$ we define

$$(4) \quad u_{z,r}(x) = \begin{cases} u(x) & \text{for } x \in U \setminus B_0(z, r), \\ \text{harmonic in } B(z, r), \text{ s.t. } u_{z,r} \text{ is continuous,} \end{cases}$$

see the figure.



Note that $\Delta u_{z,r}(x) = 0$ for $x \in B_0(z, r)$ and $u_{z,r}(x) = u(x)$ for $x \in \partial B(z, r)$. We can find $u_{z,r}$ using the formula (2).

Exercise 5. Show that if u is subharmonic in U and $B_0(z, r) \subset U$, then

- (a) $u(x) \leq u_{z,r}(x)$ for $x \in U$, and
- (b) $u_{z,r}$ is subharmonic in U .

For a finite sequence of continuous subharmonic functions $\{u_1, u_2, \dots, u_k\}$, $u_j \in C(\bar{U})$ for $j = 1, \dots, k$, define

$$v(x) = \max \{u_1(x), u_2(x), \dots, u_k(x)\}.$$

Exercise 6. Show that $v \in C(\bar{U})$ and that v is subharmonic in U .

Since g is continuous, there are constants $m \leq M$ such that

$$m = \min_{y \in \partial U} g(y) \leq g(x) \leq \max_{y \in \partial U} g(y) = M.$$

We shall show that for all balls $B(z, r) \subset U$, w_g is harmonic in $B(z, r/2)$, which implies that w_g is harmonic in U . Fix z and r so that $B(z, r) \subset U$ and choose a sequence

$\{x_k\}_{k=1}^\infty \subset B(z, r/2)$. By the definition of w_g (really by the definition of “sup”), for each k there is a sequence (of functions) $\{u_k^j\}_{j=1}^\infty \subset S_g$ such that the sequence (of numbers) $\{u_k^j(x_k)\}_{j=1}^\infty$ converges to $w_g(x_k)$,

$$w_g(x_k) = \lim_j u_k^j(x_k).$$

Exercise 7. Define

$$u^j(x) = \max \{m, u_1^j(x), u_2^j(x), \dots, u_j^j(x)\}.$$

Show that $u^j \in S_g$ and that

$$w_g(x_k) = \lim_{j \rightarrow \infty} u^j(x_k) \quad \text{for } k = 1, 2, 3, \dots$$

Explain why $m \leq u^j(x) \leq M$ for $x \in U$.

If u^j is *not* harmonic in $B(z, r)$, we can replace u^j by $u_{z,r}^j$ from equation (4). Therefore we may assume that u^j is harmonic in $B(z, r)$.

Exercise 8. Show that there is a subsequence of $\{u^j\}_{j=1}^\infty$ which converges uniformly to a function w which is harmonic in $B(z, r/2)$ and that

$$w_g(x_k) = w(x_k) \quad \text{for } k = 1, 2, 3, \dots$$

Show that w is continuous in $B(z, r/2)$.

Since w is continuous in $B(z, r/2)$, we can let $\{x_k\}_{k=1}^\infty$ be a dense countable subset and conclude that $w(x) = w_g(x)$ for all $x \in B(z, r/2)$. Now we have established that

w_g is harmonic in U .

It remains to show that w_g takes the right boundary values, i.e.,

$$(5) \quad \lim_{U \ni x \rightarrow z \in \partial U} w_g(x) = g(z).$$

In order to do this we must impose some conditions on ∂U . For each $z \in \partial U$ we define a *barrier function* Q_z which is subharmonic in U and in $C^2(\bar{U})$ such that

- (a) $Q_z(z) = 0$ and
- (b) $Q_z(x) < 0$ for $x \in \partial U \setminus \{z\}$.

A point $z \in \partial U$ is called *regular* if there exists a barrier function Q_z . The boundary ∂U is said to be regular if each point $z \in \partial U$ is regular. Choose a regular point $z \in \partial U$ and define u_- by

$$(6) \quad u_-(x) = g(z) - \epsilon + \kappa Q_z(x),$$

where ϵ and κ are positive constants. Note that u_- is subharmonic in U and that $u_- \in C(\bar{U})$. Since $g \in C(\partial U)$ we can find a δ such that

$$|g(x) - g(z)| \leq \epsilon, \quad \text{for } |x - z| \leq \delta.$$

Then we choose κ so large that

$$\kappa Q_z(x) \leq -2M \quad \text{for} \quad |x - z| > \delta,$$

$x \in \partial U$.

Exercise 9. Show that

$$u_-(x) \leq g(x) \quad \text{for} \quad x \in \partial U.$$

Next define

$$u_+(x) = g(z) + \epsilon - \kappa Q_z(x).$$

Exercise 10. Show that $-u_+$ is subharmonic (in which case u_+ is called *superharmonic*), and that $-u_+(x) \leq -g(x)$ for $x \in \partial U$. If $u \in S_g$ explain why $u(x) \leq u_+(x)$ for all $x \in U$.

Exercise 11. Show that

$$u_-(x) \leq w_g(x) \leq u_+(x),$$

for $x \in U$. Use this to show (5). (**Hint:** Let $x \rightarrow z$ and use that ϵ is arbitrary.)

We have now established

Theorem. If U is a domain with a regular boundary ∂U and $g \in C(\partial U)$, then there exists a (unique) solution u to the boundary problem (1). Furthermore $u \in C^\infty(U) \cap C(\bar{U})$.

In order to use this theorem we should find some practical (easy to use) criterion which ensures that ∂U is regular.

To this end, we say that a point $z \in \partial U$ satisfies the *external sphere condition* if there is a ball $B(y, r)$ such that

$$B_0(y, r) \cap \bar{U} = \{z\}.$$

Exercise 12. Show that if $z \in \partial U$ satisfies the external sphere condition, then z is regular. (**Hint:** Use the fundamental solution.)