

UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: MAT4301 — Partial Differential Equations

Day of examination: Tuesday, November 24th, 2020

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This problem set consists of 9 pages.

Appendices: None

Permitted aids: None.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1

a

Let $\Omega \subset \mathbb{R}^n$ be open. The mean-value formula for a function $u \in C^2(\Omega)$ satisfying $\Delta u = 0$ in Ω asserts that

$$u(x) = \oint_{\partial B(x,r)} u(y) dS(y) = \int_{B(x,r)} u(y) dy,$$

for any ball $B(x,r) \subset \Omega$, where $f_A = \frac{1}{|A|} \int_A$ for a set A of size $|A|$. Prove the first claim, namely that $u(x) = \oint_{\partial B(x,r)} u(y) dS(y)$.

Answer: For $r > 0$, define

$$\phi(r) := \oint_{\partial B(x,r)} u(y) dS(y).$$

Note that

$$\phi(0) := \lim_{r \downarrow 0} \phi(r) = \lim_{r \downarrow 0} \frac{1}{|\partial B(x,r)|} \int_{\partial B(x,r)} u(y) dS(y) = u(x).$$

Thus the claim follows if we show that $\phi'(r) = 0$.

We have

$$\phi(r) = \frac{1}{n\alpha_n r^{n-1}} \int_{\partial B(x,r)} u(y) dS(y).$$

By a change of variable,

$$z = \frac{y-x}{r}, \quad dS(z) = \frac{1}{r^{n-1}} dS(y),$$

we can write

$$\phi(r) = \frac{1}{n\alpha_n} \int_{\partial B(0,1)} u(x + rz) dS(z).$$

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This expression enables us to compute

$$\phi'(r) = \frac{1}{n\alpha_n} \int_{\partial B(0,1)} Du(x + rz) \cdot z \, dS(z).$$

Reversing the change of variable,

$$\phi'(r) = \frac{1}{n\alpha_n r^{n-1}} \int_{\partial B(x,r)} Du(y) \cdot \frac{y-x}{r} \, dS(y),$$

where we recall that $\nu(y) := \frac{y-x}{r}$ is the outward pointing unit normal vector to the ball $B(x, r)$. Hence

$$\begin{aligned} \phi'(r) &= \frac{1}{n\alpha_n r^{n-1}} \int_{\partial B(x,r)} Du(y) \cdot \nu(y) \, dS(y) \\ &= \frac{1}{n\alpha_n r^{n-1}} \int_{B(x,r)} \Delta u(y) \, dy = \frac{r}{n} \int_{B(x,r)} \Delta u(y) \, dy = 0. \end{aligned}$$

b

Let $\Omega \subset \mathbb{R}^n$ be open, and suppose $u \in C^2(\Omega)$ satisfies $\Delta u = 0$ in Ω . Use the mean-value formula to prove that

$$|u(x)| \leq \frac{C_0}{r^n} \|u\|_{L^1(B(x,r))}, \quad \forall B(x, r) \subset \Omega,$$

and

$$|Du(x)| \leq \frac{C_1}{r^{n+1}} \|u\|_{L^1(B(x,r))}, \quad \forall B(x, r) \subset \Omega,$$

where C_0 and C_1 are constants, α_n is the volume of the unit ball in $\mathbb{R}^n$, and

$$\|u\|_{L^1(B(x,r))} = \int_{B(x,r)} |u(y)| \, dy.$$

Answer: By the mean-value formula,

$$u(x) = \frac{1}{\alpha_n r^n} \int_{B(x,r)} u(y) \, dy,$$

for any ball $B(x, r) \subset \Omega$, and so

$$|u(x)| \leq \frac{1}{\alpha_n r^n} \int_{B(x,r)} |u(y)| \, dy = \frac{C_0}{r^n} \|u\|_{L^1(B(x,r))}.$$

Next, for $i = 1, \dots, n$, let us estimate u_{x_i} . Note that $\Delta u = 0$ implies that u_{x_i} solves $\Delta(u_{x_i}) = 0$, so that we can apply the mean-value formula to $u_{x_i}(x)$ and $B(x, r/2)$:

$$\begin{aligned} u_{x_i}(x) &= \frac{1}{\alpha_n (r/2)^n} \int_{B(x, r/2)} u_{x_i}(y) \, dy \\ &= \frac{2^n}{\alpha_n r^n} \int_{\partial B(x, r/2)} u(y) \nu_i(y) \, dS(y). \end{aligned}$$

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As a result,

$$\begin{aligned} |u_{x_i}(x)| &\leq \frac{2^n}{\alpha_n r^n} |\partial B(x, r/2)| \max_{\partial B(x, r/2)} |u| \\ &= \frac{2^n}{\alpha_n r^n} n \alpha_n (r/2)^{n-1} \max_{\partial B(x, r/2)} |u| = \frac{2n}{r} \max_{y \in \partial B(x, r/2)} |u(y)|. \end{aligned}$$

To control the right-hand side, we must estimate $|u(y)|$ for any y on $\partial B(x, r/2)$. Note that $y \in \partial B(x, r/2)$ implies $B(y, r/2) \subset B(x, r) \subset \Omega$. Therefore, by the mean-value formula applied to $u(y)$ and $B(y, r/2)$,

$$|u(y)| \leq \frac{1}{\alpha_n} \frac{2^n}{r^n} \|u\|_{L^1(B(y, r/2))} \leq \frac{1}{\alpha_n} \frac{2^n}{r^n} \|u\|_{L^1(B(x, r))}.$$

This bound holds for any $y \in \partial B(x, r/2)$, and so

$$\begin{aligned} |u_{x_i}(x)| &\leq \frac{2n}{r} \frac{1}{\alpha_n} \frac{2^n}{r^n} \|u\|_{L^1(B(x, r))} = \frac{2^{n+1}n}{\alpha_n} \frac{1}{r^{n+1}} \|u\|_{L^1(B(x, r))} \\ &= \frac{C_1}{r^{n+1}} \|u\|_{L^1(B(x, r))}. \end{aligned}$$

c

Suppose $u \in C^2(\mathbb{R}^n)$ is a bounded solution of $\Delta u = 0$ in $\mathbb{R}^n$, where *bounded* means that

$$|u(x)| \leq K, \quad \forall x \in \mathbb{R}^n,$$

for some constant K . Show that u must necessarily be constant.

Answer: Fix any $x \in \mathbb{R}^n$ and $r > 0$. By problem (b),

$$\begin{aligned} |Du(x)| &\leq \frac{C_1}{r^{n+1}} \int_{B(x, r)} |u(y)| \, dy \leq K \frac{C_1}{r^{n+1}} |B(x, r)| \\ &= K \frac{C_1}{r^{n+1}} \alpha_n r^n = \frac{C}{r}. \end{aligned}$$

Sending $r \rightarrow \infty$, we conclude $Du(x) = 0$. Since this holds for any $x \in \mathbb{R}^n$, the function u is constant.

d

Let Ω be a bounded open subset of $\mathbb{R}^n$. Suppose $u \in C^2(\Omega) \cap C(\overline{\Omega})$ solves the elliptic PDE

$$Lu := -\Delta u + b \cdot Du = 0 \quad \text{in } \Omega,$$

where $b = (b_1, \dots, b_n) \in \mathbb{R}^n$ is a constant vector.

(i) Establish the weak maximum principle, which asserts that

$$\min_{\partial\Omega} u \leq u(x) \leq \max_{\partial\Omega} u, \quad x \in \overline{\Omega}.$$

Hint: Develop a direct proof by showing that $u_\varepsilon(x) := u + \varepsilon v(x)$, $\varepsilon > 0$, cannot attain its maximum at an interior point of Ω , where $v(x) = e^{cx_1}$ and $c > 0$ is a constant that satisfies $c > b_1$.

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(ii) Prove that there exists at most one solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$ of the boundary value problem

$$-\Delta u + b \cdot Du = f \quad \text{in } \Omega, \quad u = g \quad \text{on } \partial\Omega,$$

where $f : \Omega \rightarrow \mathbb{R}$ and $g : \partial\Omega \rightarrow \mathbb{R}$ are given continuous functions.

Answer: (i) Notice that the function $v = e^{cx_1}$ satisfies

$$Lv = -c^2v + b_1cv = cv(b_1 - c) < 0 \quad \text{in } \Omega,$$

since $c > b_1$, and thus

$$(1) \quad Lu_\varepsilon = Lu + \varepsilon Lv = \varepsilon Lv < 0 \quad \text{in } \Omega.$$

Since u_ε is continuous in $\overline{\Omega}$, u_ε attains its maximum at some point $x_0 \in \overline{\Omega}$. If $x_0 \in \Omega$ (interior point), then by the second-derivative test,

$$Du_\varepsilon(x_0) = 0, \quad D^2u_\varepsilon(x_0) \leq 0,$$

where the latter inequality implies

$$\Delta u_\varepsilon(x_0) = \text{trace}(D^2u_\varepsilon(x_0)) \leq 0.$$

An immediate implication of the second-derivative test is that

$$Lu_\varepsilon(x_0) = -\Delta u_\varepsilon(x_0) + b \cdot Du_\varepsilon(x_0) = -\Delta u_\varepsilon(x_0) \geq 0,$$

which contradicts (1). Thus $x_0 \in \partial\Omega$ and

$$u(x) \leq \max_{\overline{\Omega}} u \leq \max_{\overline{\Omega}} u_\varepsilon = \max_{\partial\Omega} u_\varepsilon \leq \max_{\partial\Omega} u + \varepsilon \max_{\partial\Omega} v,$$

where the maximum of v over $\partial\Omega$ is a finite number. Hence, by sending $\varepsilon \rightarrow 0$, we obtain

$$u(x) \leq \max_{\partial\Omega} u, \quad x \in \overline{\Omega}.$$

Applying this result to $-u$ gives

$$-u(x) \leq \max_{\partial\Omega}(-u) = -\min_{\partial\Omega} u,$$

or

$$u(x) \geq \min_{\partial\Omega} u, \quad x \in \overline{\Omega}.$$

(ii) Let u_1, u_2 be two solutions and set $w := u_1 - u_2$. Then w solves

$$-\Delta w + b \cdot Dw = 0 \quad \text{in } \Omega, \quad w = 0 \quad \text{on } \partial\Omega.$$

By the weak maximum principle, $w \equiv 0$.

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Problem 2

a

Let Ω be a bounded open set in $\mathbb{R}^n$. Suppose that $u \in C^{2,1}(\Omega \times (0, \infty))$ satisfies the parabolic PDE

$$u_t - \Delta u + cu = 0 \quad \text{in } \Omega \times (0, \infty),$$

where $c > 0$ is a number. Let $S \in C^2(\mathbb{R})$ be a convex function satisfying

$$S(z) \geq 0, \quad zS'(z) \geq KS(z), \quad z \in \mathbb{R},$$

for some constant $K > 0$ (an example is $S(z) = z^2$). Show that $S(u)$ is a subsolution of the heat equation, that is,

$$S(u)_t - \Delta S(u) \leq 0 \quad \text{in } \Omega \times (0, \infty).$$

Answer: We compute

$$S(u)_t = S'(u)u_t, \quad DS(u) = S'(u)Du,$$

and

$$\Delta S(u) = D \cdot (S'(u)Du) = S''(u)|Du|^2 + S'(u)\Delta u.$$

Hence

$$\begin{aligned} S(u)_t - \Delta S(u) &= S'(u)(u_t - \Delta u) - S''(u)|Du|^2 \\ &= -cuS'(u) - S''(u)|Du|^2 \\ &\leq -cKS(u) - S''(u)|Du|^2 \leq 0, \end{aligned}$$

where we have also used that $S, S'' \geq 0$.

b

Let Ω be a bounded open set in $\mathbb{R}^n$. Consider the problem

$$\begin{aligned} u_t - \Delta u + cu &= f, \quad x \in \Omega, t > 0, \\ u(x, 0) &= g(x), \quad x \in \Omega, \\ u(x, t) &= h(x, t), \quad x \in \partial\Omega, t \geq 0, \end{aligned}$$

where g, h are continuous functions and $c > 0$ is a number. Use the energy method to show that there exists at most one solution $u \in C^{2,1}(\Omega \times (0, \infty))$.

Answer: Let u_1, u_2 be two solutions and set $w := u_1 - u_2$. Then w satisfies

$$\begin{aligned} w_t - \Delta w + cw &= 0, \quad x \in \Omega, t > 0, \\ w(x, 0) &= 0, \quad x \in \Omega, \\ w(x, t) &= 0, \quad x \in \partial\Omega, t \geq 0. \end{aligned}$$

In view of problem (a) with $S(z) = z^2$, we have

$$\partial_t w^2 - \Delta w^2 \leq 0.$$

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Integrating over $x \in \Omega$ yields

$$\frac{d}{dt} \int_{\Omega} w^2 dx - \int_{\partial\Omega} D(w^2) \cdot \nu dS(x) \leq 0,$$

where $\nu = \nu(x)$ is the outward-pointing unit normal vector to $\partial\Omega$. Note that

$$D(w^2) = wDw = 0 \text{ on } \partial\Omega,$$

because of the boundary condition. As a result, $\frac{d}{dt} \int_{\Omega} w^2 dx \leq 0$ and so

$$\int_{\Omega} w^2(x, t) dx \leq \int_{\Omega} w^2(x, 0) dx = 0, \quad \forall t > 0.$$

Consequently, $u_1 \equiv u_2$ in $\bar{\Omega} \times [0, \infty)$.

c

Let Ω be a bounded open set in $\mathbb{R}^n$. Consider the problem

$$\begin{aligned} u_t - \Delta u + u &= 0, & x \in \Omega, t > 0, \\ u(x, 0) &= g(x), & x \in \Omega, \\ u(x, t) &= 0, & x \in \partial\Omega, t \geq 0, \end{aligned}$$

where g is a continuous function. Use the energy method to show that for large times we have $u \approx 0$, in the sense that

$$\lim_{t \rightarrow \infty} \int_{\Omega} |u(x, t)|^2 dx = 0.$$

Answer: According to the computations in problem (a) with $S(z) = z^2$,

$$\partial_t u^2 - \Delta u^2 \leq -2u^2.$$

As before, integrating in x gives

$$\frac{d}{dt} \int_{\Omega} u^2(x, t) dx \leq -2 \int_{\Omega} u^2(x, t) dx.$$

Set $E(t) = \int_{\Omega} u^2(x, t) dx$. Then

$$E'(t) \leq -2E(t).$$

By the Gronwall inequality,

$$E(t) \leq e^{-2t} E(0),$$

which implies the desired conclusion

$$\lim_{t \rightarrow \infty} E(t) = 0.$$

(Continued on page 7.)

Problem 3

a

Consider the one-dimensional wave equation

$$(2) \quad \begin{aligned} u_{tt} - u_{xx} &= 0, & x \in \mathbb{R}, t > 0, \\ u(x, 0) &= g(x), & x \in \mathbb{R}, \\ u_t(x, 0) &= h(x), & x \in \mathbb{R}, \end{aligned}$$

where $g \in C^2(\mathbb{R})$ and $h \in C^1(\mathbb{R})$ are given functions. Use the solution formula for the one-dimensional transport equation to derive the d'Alembert solution formula for (2):

$$(3) \quad u(x, t) = \frac{1}{2}(g(x+t) + g(x-t)) + \frac{1}{2} \int_{x-t}^{x+t} h(y) dy,$$

for $(x, t) \in \mathbb{R} \times [0, \infty)$.

Answer: Writing

$$u_{tt} - u_{xx} = (\partial_t + \partial_x)(\partial_t - \partial_x)u$$

and introducing the variable $v := (\partial_t - \partial_x)u = u_t - u_x$, we obtain the transport equation

$$v_t + v_x = 0,$$

whose solution is

$$v(x, t) = v(x-t, 0) = u_t(x-t, 0) - u_x(x-t, 0) = h(x-t) - g'(x-t).$$

Going back to u , we obtain the nonhomogeneous transport equation

$$u_t - u_x = v(x, t) = h(x-t) - g'(x-t).$$

Recall that the solution to

$$w_t + bw_x = f(x, t)$$

is

$$w(x, t) = w(x-bt, 0) + \int_0^t f(x-b(t-s), s) ds.$$

Hence, with $b = -1$ and $f(x, t) = h(x-t) - g'(x-t)$,

$$\begin{aligned} u(x, t) &= u(x+t, 0) + \int_0^t h(x+(t-s)-s) - g'(x+(t-s)-s) ds \\ &= g(x+t) + \frac{1}{2} \int_0^t \frac{d}{ds} g(x+t-2s) ds + \int_0^t h(x+t-2s) ds \\ &= g(x+t) + \frac{1}{2}(g(x-t) - g(x+t)) \\ &\quad + \frac{1}{2} \int_{x-t}^{x+t} h(y) dy \quad (y := x+t-2s, dy = -2ds) \\ &= \frac{1}{2}(g(x+t) + g(x-t)) + \frac{1}{2} \int_{x-t}^{x+t} h(y) dy. \end{aligned}$$

(Continued on page 8.)

b

Prove that the function u given by the d'Alembert formula (3) is twice continuously differentiable in x and t , that is, $u \in C^2(\mathbb{R} \times [0, \infty))$. Besides, explain why $u(x, 0) = g(x)$ and $u_t(x, 0) = h(x)$ for $x \in \mathbb{R}$.

Answer: For $(x, t) \in \mathbb{R} \times [0, \infty)$, we compute

$$u_x(x, t) = \frac{1}{2}(g_x(x+t) + g_x(x-t)) + \frac{1}{2}(h(x+t) - h(x-t))$$

and

$$u_{xx}(x, t) = \frac{1}{2}(g_{xx}(x+t) + g_{xx}(x-t)) + \frac{1}{2}(h_x(x+t) - h_x(x-t)).$$

Hence u_x, u_{xx} exist and are continuous, since $g \in C^2$ and $h \in C^1$. Similarly, we find that u_t, u_{tt} exist and are continuous. As a result, we conclude that $u \in C^2$. Clearly, $u(x, 0) = g(x)$. Next, we have

$$u_t(x, t) = \frac{1}{2}(g_x(x+t) - g_x(x-t)) + \frac{1}{2}(h(x+t) + h(x-t)),$$

so that $u_t(x, 0) = h(x)$.

c

Consider the nonhomogeneous wave equation

$$(4) \quad \begin{aligned} u_{tt} - u_{xx} &= f, & x \in \mathbb{R}, t > 0, \\ u(x, 0) &= 0, & x \in \mathbb{R}, \\ u_t(x, 0) &= 0, & x \in \mathbb{R}, \end{aligned}$$

where $f \in C^2(\mathbb{R} \times [0, \infty))$ is a given function. Use Duhamel's principle to derive the solution formula

$$(5) \quad u(x, t) = \frac{1}{2} \int_0^t \int_{x-\tau}^{x+\tau} f(y, t-\tau) dy d\tau, \quad x \in \mathbb{R}, t \geq 0.$$

Verify that this function belongs to $C^2(\mathbb{R} \times [0, \infty))$, satisfies $u_{tt} - u_{xx} = f$ in $\mathbb{R} \times (0, \infty)$, and $u|_{t=0} = u_t|_{t=0} = 0$.

Answer: According Duhamel's principle, the solution of (4) is given by

$$u(x, t) = \int_0^t u(x, t; s) ds,$$

where, for each $s \geq 0$, $u(x, t; s)$ solves the homogenous wave equation

$$\begin{aligned} u_{tt} - u_{xx} &= 0, & x \in \mathbb{R}, s > t, \\ u(x, s) &= 0, & u_t|_{t=s} = f(\cdot, s), \end{aligned}$$

whose solutions is given by the d'Alembert formula (3):

$$u(x, t; s) = \frac{1}{2} \int_{x-(t-s)}^{x+(t-s)} f(y, s) dy,$$

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for $x \in \mathbb{R}$, $t \geq s$. Consequently,

$$u(x, t) = \int_0^t u(x, t; s) ds = \frac{1}{2} \int_0^t \int_{x-(t-s)}^{x+(t-s)} f(y, s) dy ds.$$

A change of variable ($\tau = t - s$, $d\tau = -ds$) gives

$$u(x, t) = \frac{1}{2} \int_0^t \int_{x-\tau}^{x+\tau} f(y, t - \tau) dy d\tau,$$

which is (5).

Let us now verify that u given by (5) is C^2 . Indeed, from the relation $u(x, t) = \int_0^t u(x, t; s) ds$ and that the solution of the homogenous wave equation is C^2 , cf. problem (b), we conclude the C^2 regularity of (5). Clearly, using (5),

$$u(x, 0) = 0,$$

and

$$u_t(x, t) = \frac{1}{2} \int_{x-t}^{x+t} f(y, 0) dy d\tau + \frac{1}{2} \int_0^t \int_{x-\tau}^{x+\tau} f_t(y, t - \tau) dy d\tau,$$

so that

$$u_t(x, 0) = 0.$$

THE END