

UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: MAT4301 — Partial Differential Equations

Day of examination: Friday, December 3, 2021

Examination hours: 15.00–19.00

This problem set consists of 9 pages.

Appendices: None

Permitted aids: None.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1

a

Let $\Omega \subset \mathbb{R}^n$ be open. For $r > 0$, define

$$\phi(r) := \frac{1}{n\alpha_n r^{n-1}} \int_{\partial B(x,r)} u(y) dS(y), \quad B(x,r) \subset \Omega,$$

where $B(x,r) = \{y \in \mathbb{R}^n : |y-x| \leq r\}$ is the (closed) ball with center x and radius $r > 0$, and α_n is the volume of the unit ball in $\mathbb{R}^n$. Show that

$$\phi'(r) = \frac{1}{n\alpha_n r^{n-1}} \int_{B(x,r)} \Delta u(y) dy.$$

Answer: By a change of variable,

$$z = \frac{y-x}{r}, \quad dS(z) = \frac{1}{r^{n-1}} dS(y),$$

we can write

$$\phi(r) = \frac{1}{n\alpha_n} \int_{\partial B(0,1)} u(x+rz) dS(z).$$

This allows us to compute

$$\phi'(r) = \frac{1}{n\alpha_n} \int_{\partial B(0,1)} Du(x+rz) \cdot z dS(z).$$

Reversing the change of variable,

$$\phi'(r) = \frac{1}{n\alpha_n r^{n-1}} \int_{\partial B(x,r)} Du(y) \cdot \frac{y-x}{r} dS(y),$$

(Continued on page 2.)

where we recall that $\nu(y) := \frac{y-x}{r}$ is the outward pointing unit normal vector to the ball $B(x, r)$. Integration by parts gives

$$\begin{aligned}\phi'(r) &= \frac{1}{n\alpha_n r^{n-1}} \int_{\partial B(x,r)} Du(y) \cdot \nu(y) dS(y) \\ &= \frac{1}{n\alpha_n r^{n-1}} \int_{B(x,r)} \Delta u(y) dy.\end{aligned}\tag{1}$$

b

Let $\Omega \subset \mathbb{R}^n$ be an open set. Suppose $u \in C^2(\Omega)$ is a function satisfying $u(x) = \frac{1}{n\alpha_n r^{n-1}} \int_{\partial B(x,r)} u(y) dS(y)$, $\forall B(x, r) \subset \Omega$. Prove that $\Delta u = 0$ in Ω .

Answer: Suppose there exists $x \in \Omega$ such that $\Delta u(x) \neq 0$, say $\Delta u(x) > 0$. By continuity of Δu , there exists a ball $B(x, r) \subset \Omega$ such that $\Delta u > 0$ in $B(x, r)$. Recalling (1), we obtain

$$\phi'(r) = \frac{1}{n\alpha_n r^{n-1}} \int_{B(x,r)} \Delta u(y) dy > 0,$$

contradicting the assumption of the problem, namely that $\phi(r) = u(x)$ for all $r > 0 \implies \phi'(r) = 0$.

c

Let $\Omega \subset \mathbb{R}^n$ be a bounded, open and connected set. We say that $u \in C^2(\Omega)$ is a *subsolution* (of the Laplace equation) if $\Delta u \geq 0$ in Ω . Similarly, we say that u is a *supersolution* (of the Laplace equation) if $\Delta u \leq 0$ in Ω .

Prove the following maximum principle: if $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is a subsolution, then

$$u(x) \leq \max_{\partial\Omega} u, \quad x \in \Omega.$$

Hint: Develop a direct argument by showing that $u_\varepsilon(x) := u(x) + \varepsilon|x|^2$ cannot attain its maximum at an interior point of Ω .

Answer: (i) Since u is a subsolution ($\Delta u \geq 0$ in Ω), it follows that $\Delta u_\varepsilon = \Delta u + n\varepsilon \geq n\varepsilon > 0$. Since u_ε is continuous in $\overline{\Omega}$, u_ε attains its maximum at some point $x_0 \in \overline{\Omega}$. Suppose $x_0 \in \Omega$ (interior point). By the second-derivative test at an interior maximum, $D^2 u_\varepsilon(x_0) \leq 0$ and thus $\Delta u_\varepsilon(x_0) = \text{trace}(D^2 u_\varepsilon(x_0)) \leq 0$, a contradiction. Thus, $x_0 \in \partial\Omega$ and

$$u(x) \leq \max_{\overline{\Omega}} u \leq \max_{\overline{\Omega}} u_\varepsilon = \max_{\partial\Omega} u_\varepsilon \leq \max_{\partial\Omega} u + \varepsilon \max_{x \in \partial\Omega} |x|.$$

Sending $\varepsilon \rightarrow 0$, we obtain $u(x) \leq \max_{\partial\Omega} u$.

d

The so-called *comparison principle* states that, for any subsolution $u \in C^2(\Omega) \cap C(\overline{\Omega})$ and any supersolution $v \in C^2(\Omega) \cap C(\overline{\Omega})$ such that $u \leq v$ on the boundary $\partial\Omega$, one has $u \leq v$ in Ω .

Use (c) to prove the comparison principle.

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Answer: Set $w := u - v$, where u is a subsolution and v is a supersolution. Then $\Delta w = \Delta u - \Delta v \geq 0$ in Ω , i.e., w is a subsolution as well. Moreover, observe that $w \leq 0$ on $\partial\Omega$. By the maximum principle for subsolutions (i),

$$w(x) \leq \max_{\partial\Omega} w \leq 0, \quad x \in \Omega.$$

This concludes the proof of the comparison principle.

Problem 2

If Ω is a bounded open subset of $\mathbb{R}^n$, then we denote by $\Omega_T = \Omega \times (0, T]$ the parabolic cylinder and by $\Gamma_T = \overline{\Omega_T} \setminus \Omega_T$ the parabolic boundary (comprising the bottom and vertical sides of $\Omega \times [0, T]$).

a

Suppose $\Omega \subset \mathbb{R}^n$ is bounded, open and connected. A function $u \in C^{2,1}(\Omega_T)$ is called a supersolution of the heat equation if

$$u_t - \Delta u \geq 0 \quad \text{in } \Omega_T.$$

It is known that a supersolution u satisfies the mean-value property

$$u(x, t) \geq \frac{1}{4r^n} \iint_{E(x, t; r)} u(y, s) \frac{|x - y|^2}{(t - s)^2} dy ds, \quad (2)$$

for any heat ball $E(x, t; r) \subset \Omega_T$ with top center at (x, t) and radius $r > 0$. In what follows, you can use freely that $\iint_{E(x, t; r)} \frac{|x - y|^2}{(t - s)^2} dy ds = 4r^n$.

Let u be a supersolution of the heat equation, and suppose $(x_0, t_0) \in \Omega_T$ is such that

$$u(x_0, t_0) = \min_{\Omega_T} u.$$

Prove that

$$u = \text{constant in } \Omega_{t_0} := \Omega \times (0, t_0].$$

Answer: For small enough r , $E(x_0, t_0; r) \subset \Omega_T$. Thus, by (2),

$$\begin{aligned} m := u(x_0, t_0) &\geq \frac{1}{4r^n} \iint_{E(x_0, t_0; r)} u(y, s) \frac{|x_0 - y|^2}{(t_0 - s)^2} dy ds \\ &\geq m \frac{1}{4r^n} \iint_{E(x_0, t_0; r)} \frac{|x_0 - y|^2}{(t_0 - s)^2} dy ds = m, \end{aligned}$$

so that $\frac{1}{4r^n} \iint_{E(x_0, t_0; r)} (u(y, s) - m) \frac{|x_0 - y|^2}{(t_0 - s)^2} dy ds = 0$. This (and $u \geq m$) necessarily implies that $u = m$ in $E(x_0, t_0; r)$.

Let $(y_0, s_0) \in \Omega_T$, $s < t_0$, be a point that does not belong to $E(x_0, t_0; r)$ but is such that the line segment L connecting (x_0, t_0) to (y_0, s_0) belongs to Ω_T . Covering L by a finite number of heat balls, we can apply the previous argument repeatedly to conclude that $u = m$ on L . More generally, by the connectivity of Ω , any point y_0 can be connected to x_0 by a finite number of line segments, and on each line segment $u = m$. Consequently,

$$u(y, s) = u(x_0, t_0) = m, \quad \text{for all } y \in \Omega \text{ and } s < t_0.$$

(Continued on page 4.)

b

Suppose $\Omega \subset \mathbb{R}^n$ is bounded, open.

(i) Let $u \in C^{2,1}(\Omega_T) \cap C(\overline{\Omega_T})$ solve the parabolic equation

$$u_t - \Delta u + b(x, t) \cdot Du = 0 \quad \text{in } \Omega_T, \quad (3)$$

where $b = b(x, t) = (b_1(x, t), \dots, b_n(x, t))$ is a vector of continuous functions. Establish the weak maximum principle, which asserts that

$$\min_{\Gamma_T} u \leq u(x, t) \leq \max_{\Gamma_T} u, \quad (x, t) \in \overline{\Omega_T}.$$

Hint: Show that $u_\varepsilon := u - \varepsilon t$, $\varepsilon > 0$, cannot attain its maximum at an interior point in Ω_T .

(ii) Let $g \in C(\Gamma_T)$, $f \in C(\Omega_T)$ and $b_i \in C(\Omega_T)$, $i = 1, \dots, n$, and set $b := (b_1, \dots, b_n)$. Prove that there exists at most one solution $u \in C^{2,1}(\Omega_T) \cap C(\overline{\Omega_T})$ of the initial-boundary value problem

$$\begin{aligned} u_t - \Delta u + b(x, t) \cdot Du &= f \quad \text{in } \Omega_T, \\ u &= g \quad \text{on } \Gamma_T. \end{aligned} \quad (4)$$

Answer: (i) Since u solves (3),

$$\partial_t u_\varepsilon - \Delta u_\varepsilon + b \cdot Du_\varepsilon = -\varepsilon + \partial_t u - \Delta u + b \cdot Du = -\varepsilon < 0. \quad (5)$$

Since u_ε is continuous on $\overline{\Omega_T}$ and $\overline{\Omega_T}$ is compact (bounded and closed), there exists a point $(x_0, t_0) \in \overline{\Omega_T}$ such that $\max_{\overline{\Omega_T}} u_\varepsilon = u_\varepsilon(x_0, t_0)$. If $(x_0, t_0) \in \Omega_T$ (interior maximum point), then the derivative test gives

$$\partial_t u_\varepsilon(x_0, t_0) = 0, \quad D_x u_\varepsilon(x_0, t_0) = 0, \quad \Delta_x u_\varepsilon(x_0, t_0) \leq 0,$$

which implies that

$$\partial_t u_\varepsilon(x_0, t_0) - \Delta u_\varepsilon(x_0, t_0) + b \cdot Du_\varepsilon(x_0, t_0) = -\Delta u_\varepsilon(x_0, t_0) \geq 0.$$

This is a contradiction to (5). Hence $(x_0, t_0) \in \Gamma_T$ and

$$u_\varepsilon(x, t) \leq \max_{\overline{\Omega_T}} u_\varepsilon = \max_{\Gamma_T} u_\varepsilon,$$

which implies that

$$u(x, t) \leq \max_{\overline{\Omega_T}} u \leq \max_{\overline{\Omega_T}} u_\varepsilon + \varepsilon T = \max_{\Gamma_T} u_\varepsilon + T\varepsilon \leq \max_{\Gamma_T} u + T\varepsilon,$$

where we have used that

$$\max_{\Gamma_T} u_\varepsilon \leq \max_{\Gamma_T} u + \max_{\Gamma_T} (-\varepsilon t) = \max_{\Gamma_T} u - \varepsilon \min_{[0, T]} t = \max_{\Gamma_T} u.$$

Sending $\varepsilon \rightarrow 0$ yields

$$u(x, t) \leq \max_{\Gamma_T} u, \quad (x, t) \in \overline{\Omega_T}.$$

(Continued on page 5.)

Applying this result to $-u$ gives

$$(-u)(x, t) \leq \max_{\Gamma_T}(-u) = -\min_{\Gamma_T} u$$

or

$$u(x, t) \geq \min_{\Gamma_T} u, \quad (x, t) \in \overline{\Omega_T}.$$

(ii) Suppose u_1, u_2 are two solutions of (4) and set $u := u_1 - u_2$. Then

$$u_t - \Delta u + b \cdot Du = 0 \quad \text{in } \Omega_T, \quad u = 0 \text{ on } \Gamma_T.$$

By part (i), for any $(x, t) \in \Omega_T$,

$$u(x, t) \leq \max_{\Gamma_T} u = 0.$$

Similarly, for any $(x, t) \in \Omega_T$,

$$u(x, t) \geq \min_{\Gamma_T} u = 0.$$

Hence $u \equiv 0$.

c

Suppose $\Omega \subset \mathbb{R}^n$ is bounded, open. Let $u \in C^{2,1}(\Omega_T)$ solve

$$u_t - \Delta u + b(x, t) \cdot Du = 0 \quad \text{in } \Omega_T, \quad (6)$$

where $b = b(x, t) = (b_1(x, t), \dots, b_n(x, t))$ is a vector of continuous functions for which $\operatorname{div} b(x, t) = \sum_{i=1}^n \partial_{x_i} b_i(x, t) = 0$ for all $(x, t) \in \Omega_T$.

Let $S \in C^2(\mathbb{R})$ be a positive function (e.g., $S(z) = z^2/2$). Show that $S(u(x, t))$ satisfies

$$S(u)_t - \Delta S(u) + \operatorname{div}(b(x, t)S(u)) \leq 0 \quad \text{in } \Omega_T, \quad (7)$$

where $\operatorname{div}(b(x, t)S(u)) = \sum_{i=1}^n \partial_{x_i}(b_i(x, t)S(u))$.

Answer: We compute

$$S(u)_t = S'(u)u_t, \quad DS(u) = S'(u)Du,$$

and

$$\Delta S(u) = D \cdot (S'(u)Du) = S''(u)|Du|^2 + S'(u)\Delta u.$$

Using these relations, we obtain

$$\begin{aligned} S(u)_t - \Delta S(u) + b(x, t) \cdot DS(u) &= S'(u) \underbrace{(u_t - \Delta u + b(x, t) \cdot Du)}_{=0} - S''(u)|Du|^2 \\ &= -S''(u)|Du|^2 \leq 0, \end{aligned}$$

where we have used that S is convex ($S'' \geq 0$). Finally, we utilise the divergence-free condition on b to write

$$b(x, t) \cdot DS(u) = \operatorname{div}(b(x, t)S(u)).$$

This establishes (7).

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d

Suppose $\Omega \subset \mathbb{R}^n$ is bounded and open, with outward unit normal vector ν . Consider the initial-boundary value problem

$$\begin{aligned} u_t - \Delta u + b(x, t) \cdot Du &= f \quad \text{in } \Omega_T, \\ u(x, 0) &= u_0(x), \quad x \in \Omega, \\ u(x, t) &= h(x, t), \quad x \in \partial\Omega, t \geq 0, \end{aligned} \tag{8}$$

where $u_0, h, b = (b_1, \dots, b_n)$ are continuous functions, and $\operatorname{div} b(x, t) = 0$ for all $(x, t) \in \Omega_T$. Let u and v be two classical solutions of (8) with initial data u_0 and v_0 , respectively. Use (c) to establish the L^2 stability estimate

$$\int_{\Omega} |u(x, t) - v(x, t)|^2 dx \leq \int_{\Omega} |u_0(x) - v_0(x)|^2 dx, \quad t > 0.$$

Answer: Set $w := u - v$ and $w_0 := u_0 - v_0$. Then w solves the problem

$$\begin{aligned} w_t - \Delta w + b(x, t) \cdot Dw &= 0, \quad x \in \Omega, t > 0, \\ w(x, 0) &= w_0(x), \quad x \in \Omega, \\ w(x, t) &= 0, \quad x \in \partial\Omega, t \geq 0. \end{aligned}$$

We can apply the previously established inequality (7) with $S(z) = \frac{1}{2}z^2$. Integrating the resulting inequality supplies

$$\frac{d}{dt} \int_{\Omega} \frac{w^2}{2} dx - \int_{\Omega} \Delta \left(\frac{w^2}{2} \right) dx + \int_{\Omega} \operatorname{div} \left(b(x, t) \frac{w^2}{2} \right) dx \leq 0.$$

Using the boundary condition $w|_{\Omega} = 0$, it follows that

$$\int_{\Omega} \operatorname{div} \left(b(x, t) \frac{w^2}{2} \right) dx = \int_{\partial\Omega} \frac{w^2}{2} b(x, t) \cdot \nu(x) dS(x) = 0$$

and, using also the chain rule,

$$\int_{\Omega} \Delta \left(\frac{w^2}{2} \right) dx = \int_{\partial\Omega} D \left(\frac{w^2}{2} \right) \cdot \nu dS(x) = \int_{\partial\Omega} w Dw \cdot \nu dS(x) = 0.$$

Consequently,

$$\frac{d}{dt} \int_{\Omega} w^2 dx \leq 0,$$

which implies that

$$\int_{\Omega} w^2(t, x) dx \leq \int_{\Omega} w_0^2(x) dx,$$

and, recalling the definitions of w, w_0 , this proves the claim.

(Continued on page 7.)

Problem 3

a

Consider the initial-value problem for the one-dimensional wave equation:

$$\begin{aligned} u_{tt} - u_{xx} &= 0 \quad \text{in } \mathbb{R} \times (0, \infty), \\ u(x, 0) &= 0, \quad u_t(x, 0) = h(x), \quad x \in \mathbb{R}, \end{aligned}$$

where h is a continuous function.

(i) Use the change of variables

$$\xi = x + t, \eta = x - t \iff x = \frac{\xi + \eta}{2}, t = \frac{\xi - \eta}{2}$$

to show that $u_{tt} - u_{xx} = 0$ if and only if the function

$$v(\xi, \eta) = u\left(\frac{\xi + \eta}{2}, \frac{\xi - \eta}{2}\right)$$

satisfies the PDE $v_{\xi\eta} = 0$.

(ii) Explain that the general solution of $v_{\xi\eta} = 0$ is $v(\xi, \eta) = F(\xi) + G(\eta)$, for arbitrary (continuously differentiable) functions F, G .

(iii) Use (i) and (ii) to derive the d'Alembert solution formula

$$u(x, t) = \frac{1}{2} \int_{x-t}^{x+t} h(y) dy.$$

Answer: (i) Differentiating gives

$$v_{\xi} = \frac{1}{2}u_x + \frac{1}{2}u_t,$$

and

$$v_{\xi\eta} = \frac{1}{4}u_{xx} - \frac{1}{4}u_{xt} + \frac{1}{4}u_{tx} - \frac{1}{4}u_{tt} = -\frac{1}{4}(u_{tt} - u_{xx}) = 0.$$

(ii) Integrating the equation $v_{\xi\eta} = 0$ twice, first w.r.t. η and then w.r.t. ξ , it follows that $v(\xi, \eta) = F(\xi) + G(\eta)$ for some functions F, G . On the other hand, by differentiating the expression $v(\xi, \eta) = F(\xi) + G(\eta)$ twice, we see that $v_{\xi\eta} = 0$.

(iii) We have that $u(x, t) = u\left(\frac{\xi+t}{2}, \frac{\xi-t}{2}\right) = v(\xi, \eta)$ and so, by (ii),

$$u(x, t) = F(\xi) + G(\eta) = F(x + t) + G(x - t).$$

Using the initial condition $u(x, 0) = 0$, we deduce

$$F(x) + G(x) = 0.$$

Next, using the other initial condition — $u_t(x, 0) = h(x)$,

$$F'(x) - G'(x) = h(x).$$

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These two conditions imply that $2F'(x) = h(x)$ or, integrating from 0 to x ,

$$F(x) = F(0) + \frac{1}{2} \int_0^x h(y) dy.$$

Consequently,

$$\begin{aligned} u(x, t) &= F(x+t) + G(x-t) \\ &= F(x+t) - F(x-t) = \frac{1}{2} \int_{x-t}^{x+t} h(y) dy. \end{aligned}$$

b

Let $u \in C^2(\Omega_T)$ satisfy the one-dimensional wave equation

$$u_{tt} - u_{xx} = 0 \quad \text{in } \mathbb{R} \times (0, \infty).$$

Fix $x_0 \in \mathbb{R}$, $t_0 > 0$. Introduce the set

$$K(x_0, t_0) = \{(x, t) \in \mathbb{R}^2 : x \in [x_0 - (t_0 - t), x_0 + t_0 - t], t \in [0, t_0]\},$$

often referred to as the backwards wave cone with apex (x_0, t_0) . Establish the following *finite speed of propagation* property: $u(x, 0) = u_t(x, 0) = 0$ for all $x \in [x_0 - t_0, x_0 + t_0]$ implies that $u = 0$ in $K(x_0, t_0)$.

Hint: Use the local energy function

$$e(t) = \frac{1}{2} \int_{x_0-t_0+t}^{x_0+t_0-t} u_t^2(x, t) + u_x^2(x, t) dx, \quad \text{for } t \in [0, t_0].$$

Answer: We compute

$$\begin{aligned} e'(t) &= \int_{x_0-t_0+t}^{x_0+t_0-t} u_t u_{tt} + u_x u_{xt} dx \\ &\quad - \frac{1}{2} u_t^2(x_0 + t_0 - t, t) - \frac{1}{2} u_x^2(x_0 + t_0 - t, t) \\ &\quad - \frac{1}{2} u_t^2(x_0 - t_0 + t, t) - \frac{1}{2} u_x^2(x_0 - t_0 + t, t) \\ &=: \int_{x_0-t_0+t}^{x_0+t_0-t} u_t u_{tt} + u_x u_{xt} dx + A. \end{aligned}$$

An integration by parts (in x) gives

$$\int_{x_0-t_0+t}^{x_0+t_0-t} u_x u_{xt} dx = B - \int_{x_0-t_0+t}^{x_0+t_0-t} u_{xx} u_t dx,$$

where the boundary terms are

$$\begin{aligned} B &= u_x(x_0 + t_0 - t, t) u_t(x_0 + t_0 - t, t) \\ &\quad - u_x(x_0 - t_0 + t, t) u_t(x_0 - t_0 + t, t). \end{aligned}$$

Clearly, using the basic inequality $ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2$ for all $a, b \in \mathbb{R}$,

$$\begin{aligned} B &\leq \frac{1}{2} u_x^2(x_0 + t_0 - t, t) + \frac{1}{2} u_t^2(x_0 + t_0 - t, t) \\ &\quad + \frac{1}{2} u_x^2(x_0 - t_0 + t, t) + \frac{1}{2} u_t^2(x_0 - t_0 + t, t), \end{aligned}$$

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which implies that $A + B \leq 0$ and thus

$$e'(t) = \int_{x_0-t_0+t}^{x_0+t_0-t} u_t \underbrace{(u_{tt} - u_{xx})}_{=0} dx + A + B \leq 0.$$

Since $e(0) = 0$, this implies $e(t) = 0$ for all $t \in [0, t_0]$, i.e., $u_t = 0$ and $Du = 0$, and so $u = \text{constant} = 0$ in $K(x_0, t_0)$.

THE END