

UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: MAT4301 — Partial differential equations

Day of examination: 18 December 2023

Examination hours: 09:00 – 13:00

This problem set consists of 6 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Note:

- You can get a total of 100 points. The point distribution is specified for each problem.
- All answers must be justified.

Corrections to solution in red

Problem 1 (25p)

Consider the heat equation

$$\begin{cases} u_t = \Delta u & \text{for } x \in \mathbb{R}^n, t > 0 \\ u(x, 0) = g(x) & \text{for } x \in \mathbb{R}^n \end{cases} \quad (1)$$

for some continuous, bounded function $g: \mathbb{R}^n \rightarrow \mathbb{R}$.

1a (15p)

Write down the formula for the solution of (1), and use this formula to show that

$$\inf_{\mathbb{R}^n} g \leq u(x, t) \leq \sup_{\mathbb{R}^n} g \quad \text{for all } x \in \mathbb{R}^n, t > 0.$$

1b (10p)

How “smooth” (or “regular”) can we expect the above solution u to be? Justify your claim.

Solution:

(Continued on page 2.)

1a

The solution formula is

$$u(x, t) = \int_{\mathbb{R}^n} \Phi(x - y, t) g(y) dy = \frac{1}{\sqrt{4\pi t}^n} \int_{\mathbb{R}^n} e^{-|x-y|^2/4t} g(y) dy.$$

Since $\Phi(z, t) > 0$ and $\int_{\mathbb{R}^n} \Phi(z, t) dz = 1$ for all $t > 0$, we find that

$$u(x, t) \leq \int_{\mathbb{R}^n} \Phi(x - y, t) \left(\sup_{\mathbb{R}^n} g \right) dy = \sup_{\mathbb{R}^n} g,$$

and likewise

$$u(x, t) \geq \int_{\mathbb{R}^n} \Phi(x - y, t) \left(\inf_{\mathbb{R}^n} g \right) dy = \inf_{\mathbb{R}^n} g.$$

1b

We can expect u to be C^∞ . Indeed, for any multiindex $\alpha \in \mathbb{N}_0^{n+1}$ we have

$$D^\alpha u(x, t) = \int_{\mathbb{R}^d} D^\alpha \Phi(x - y, t) g(y) dy.$$

The derivatives of Φ are of the form

$$D^\alpha \Phi(z, t) = \frac{p_\alpha(z, t)}{q_\alpha(\sqrt{t})} e^{-|z|^2/4t}$$

for polynomials p_α and q_α , and where $q_\alpha(\tau) > 0$ for all $\tau > 0$. Since the exponential $e^{-|z|^2/4t}$ goes to zero as $|z| \rightarrow \infty$ much faster than any polynomial can go to ∞ , these derivatives are integrable, and therefore $D^\alpha u(x, t)$ is continuous and well-defined.

Problem 2 (50p)

Let $\Omega \subset \mathbb{R}^n$ be open, bounded and connected. A function $u \in C^2(\Omega)$ is a *subsolution* of Laplace's equation if $\Delta u(x) \geq 0$ for all $x \in \Omega$.

2a (10p)

Find an example of $\Omega \subset \mathbb{R}^3$ and two *different* subsolutions u, v of Laplace's equation on Ω satisfying $u = v = 0$ on $\partial\Omega$. You can for instance let $\Omega = B(0, 1)$.

2b (15p)

Let u be a subsolution of Laplace's equation on some domain Ω . For a fixed $x^0 \in \Omega$, define $\varphi(r) := \int_{B(x^0, r)} u(y) dy$ (for $r > 0$ small enough so that $B(x^0, r) \subset \Omega$). Show that $\varphi'(r) \geq 0$ for all r .

Hint: You might need to use both change of variables, polar coordinates, and the divergence theorem.

(Continued on page 3.)

2c (10p)

Use Problem 2b to prove the strong maximum principle: If $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is a subsolution of Laplace's equation, and there is a point $x^0 \in \Omega$ where $u(x^0) = M := \max_{\overline{\Omega}} u$, then u is constant.

2d (15p)

Let $u_1, u_2 \in C^2(\Omega) \cap C(\overline{\Omega})$ be solutions to

$$\begin{cases} \Delta u_1 = f_1 & \text{in } \Omega \\ u_1 = g_1 & \text{on } \partial\Omega, \end{cases} \quad \begin{cases} \Delta u_2 = f_2 & \text{in } \Omega \\ u_2 = g_2 & \text{on } \partial\Omega, \end{cases} \quad (2)$$

where $f_1, f_2 \in C(\Omega)$ and $g_1, g_2 \in C(\partial\Omega)$ satisfy $f_1 \leq f_2$ and $g_1 \geq g_2$. Use Problem 2c to show that $u_1 \geq u_2$.

Solution:**2a**

We can for instance pick $\Omega = B(0, 1)$, $u(x) = 0$ and $v(x) = |x|^2 - 1$.

2b

We compute

$$\begin{aligned} \varphi'(r) &= \frac{d}{dr} \left(\frac{1}{|B(x^0, r)|} \int_{B(x^0, r)} u(y) dy \right) = \frac{d}{dr} \left(\frac{1}{\alpha_n} \int_{B(0,1)} u(x^0 + rz) dz \right) \\ &= \frac{1}{\alpha_n} \int_{B(0,1)} Du(x^0 + rz) \cdot z dz = \frac{1}{\alpha_n} \int_0^1 \int_{\partial B(0,s)} Du(x^0 + rz) \cdot z dS(z) ds \\ &= \frac{1}{\alpha_n} \int_0^1 s \int_{\partial B(0,s)} Du(x^0 + rz) \cdot \nu(z) dS(z) ds \\ &= \frac{1}{\alpha_n} \int_0^1 s \int_{B(0,s)} \operatorname{div}_z (Du(x^0 + rz)) dz ds \\ &= \frac{1}{\alpha_n} \int_0^1 s \int_{B(0,s)} r \Delta u(x^0 + rz) dz ds \\ &\geq 0 \end{aligned}$$

(where $\alpha_n = |B(0, 1)|$).

2c

Assume that there is a point x^0 where $u(x^0) = M$. Then for any small $r > 0$,

$$M = u(x^0) = \lim_{s \rightarrow 0} \varphi(s) \leq \varphi(r) = \int_{B(x^0, r)} u(y) dy \leq M,$$

so necessarily there is equality above. But then u must be constantly equal to M in $B(x^0, r)$.

(Continued on page 4.)

Let $U = \{x \in \bar{\Omega} : u(x) = M\}$, a non-empty set. Then U is closed (since u is continuous), but by the above argument, U is also open (since for every $x \in U$, there is some $r > 0$ such that $B(x, r) \subset U$). The only non-empty subsets of a connected set Ω which are both closed and open, is Ω itself. Hence, u is constant.

2d

Define $w = u_2 - u_1$, which satisfies $\Delta w = f_2 - f_1 \geq 0$, so it is a subsolution of Laplace's equation. Let $x^0 \in \bar{\Omega}$ be a point where $w(x^0) = \max_{\bar{\Omega}} w$. If $x^0 \in \partial\Omega$ then

$$\max_{\bar{\Omega}} w = w(x^0) = g_2(x^0) - g_1(x^0) \leq 0,$$

so $w \leq 0$, that is, $u_1 \geq u_2$. If $x^0 \in \Omega$ then, by Problem 2c, w must be constant, so at all points $x \in \bar{\Omega}$,

$$w(x) = w|_{\partial\Omega} = g_2 - g_1 \leq 0,$$

so again, $w \leq 0$, that is, $u_1 \geq u_2$.

Problem 3 (15p)

Consider the conservation law

$$\begin{cases} u_t + f(u)_x = 0 \\ u(x, 0) = g(x). \end{cases} \quad (3)$$

Use the method of characteristics to find a weak solution of (3) when $f(u) = u^2$ and

$$g(x) = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } 0 \leq x \leq 1 \\ 1 & \text{if } 1 < x. \end{cases}$$

Sketch the characteristic curves in the x - t -plane.

Solution: We use the method of characteristics. The characteristics satisfy $\dot{x}(t) = f'(u(x(t), t))$, and the solution is constant along $x(t)$, so

$$x(t) = x^0 + tf'(g(x^0)) = x^0 + 2tg(x^0) = \begin{cases} x^0 & \text{if } x^0 < 0 \\ x^0(1 + 2t) & \text{if } 0 \leq x^0 \leq 1 \\ x^0 + 2t & \text{if } 1 < x^0. \end{cases}$$

Note that none of these curves collide for $t > 0$, and that they cover the entire upper half plane. For an arbitrary point $(x, t) \in \mathbb{R} \times (0, \infty)$, we have $x = x(t)$ if and only if

$$x^0 = \begin{cases} x & \text{if } x < 0 \\ x/(1 + 2t) & \text{if } 0 \leq x \leq 1 + 2t \\ x - 2t & \text{if } 1 + 2t < x. \end{cases}$$

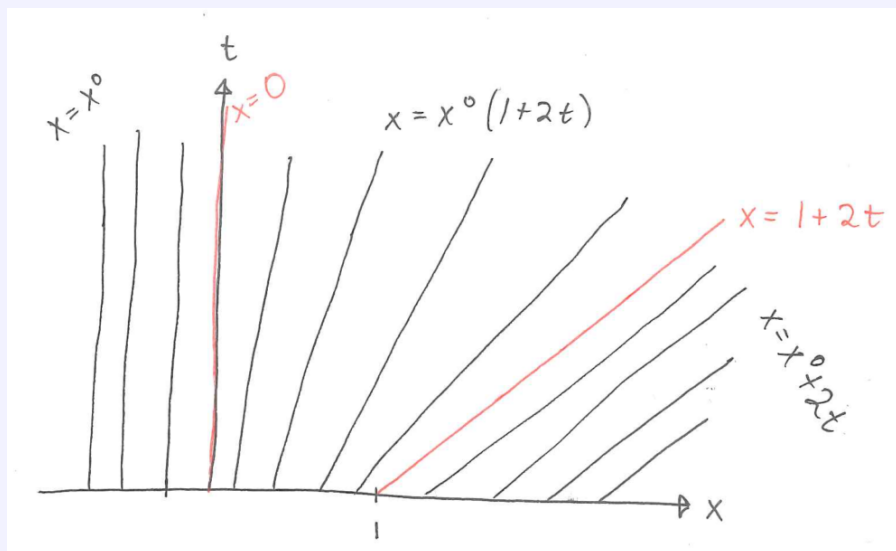
(Continued on page 5.)

This leads to the solution

$$u(x, t) = g(x^0) = \begin{cases} 0 & \text{if } x < 0 \\ x/(1+2t) & \text{if } 0 \leq x \leq 1+2t \\ 1 & \text{if } 1+2t < x. \end{cases}$$

This is a Lipschitz continuous solution without discontinuities, so it automatically satisfies the entropy condition.

The characteristics look as follows in the x - t -plane:



Problem 4 (10p)

Consider the Hamilton–Jacobi equation

$$\begin{cases} u_t + \frac{(u_x)^2}{2} = 0 & \text{for } x \in \mathbb{R}, t > 0 \\ u(x, 0) = |x| & \text{for } x \in \mathbb{R}. \end{cases} \quad (4)$$

Prove that this problem satisfies the conditions necessary to apply the Hopf–Lax formula. Then use this formula to deduce the following expression for the solution:

$$u(x, t) = \begin{cases} |x| - \frac{t}{2} & \text{if } |x| > t \\ \frac{x^2}{2t} & \text{if } |x| \leq t. \end{cases} \quad (5)$$

Solution: The Hamiltonian $H(u) = \frac{u^2}{2}$ is convex and grows superlinearly, and the initial data $g(x) = |x|$ is Lipschitz continuous, so we may use the Hopf–Lax formula. This formula gives the solution

$$u(x, t) = \min_{y \in \mathbb{R}} \left(g(y) + \frac{1}{t} L((x - y)/t) \right)$$

where $L = H^*$ is the Legendre dual of H . We get

$$L(v) = \sup_p \left(pv - p^2/2 \right) = v^2/2,$$

(Continued on page 6.)

so

$$u(x, t) = \min_{y \in \mathbb{R}} \left(|y| + \frac{(x - y)^2}{2t} \right).$$

The minimum of the above expression is either attained at a critical point of the function being minimized, or at the non-differentiable point $y = 0$. At $y = 0$ we get

$$\left(|y| + \frac{(x - y)^2}{2t} \right) \Big|_{y=0} = \frac{x^2}{2t}.$$

The critical points are found as follows:

$$0 = \frac{d}{dy} \left(|y| + \frac{(x - y)^2}{2t} \right) = \text{sign}(y) + \frac{y - x}{t}$$

so we get

$$y = \begin{cases} x - t & \text{if } x - t > 0 \\ x + t & \text{if } x + t < 0. \end{cases}$$

At each of these critical points we have

$$\left(|y| + \frac{(x - y)^2}{2t} \right) \Big|_{y=x \pm t} = |x \pm t| + \frac{t}{2}.$$

Thus,

$$u(x, t) = \min \left(|x \pm t| + \frac{t}{2}, \frac{x^2}{2t} \right).$$

Studying each of these expressions, we see that they coincide when $x = t$, so

$$u(x, t) = \begin{cases} -x - \frac{t}{2} & \text{if } x < -t \\ \frac{x^2}{2t} & \text{if } -t \leq x \leq t \\ x - \frac{t}{2} & \text{if } x > t \end{cases} = \begin{cases} |x| - \frac{t}{2} & \text{if } |x| > t \\ \frac{x^2}{2t} & \text{if } |x| \leq t. \end{cases}$$

THE END