

MAT4305 – Partial differential equations and Sobolev spaces I

Mandatory assignment 1 of 1

Submission deadline

Thursday 13 April 2023 at 14:30 in Canvas (canvas.uio.no).

Instructions

You can choose between writing in English or Norwegian.

You can choose between scanning handwritten notes or typing the solution directly on a computer (for instance with $\text{\LaTeX}$). The assignment must be submitted as a single PDF. Scanned pages must be clearly legible; please use either the “Color” or “Grayscale” settings. The submission must contain your name, course and assignment number.

It is expected that you give a clear presentation with all necessary explanations. Remember to include all relevant plots and figures. As a rule of thumb, you need to correctly answer $2/3$ of the assignment in order to get it passed, and to show that you have made an attempt at solving all of the problems.

All aids, including collaboration, are allowed, but the submission must be written by you and reflect your understanding of the subject. If we doubt that you have understood the content you have handed in, we may request that you give an oral account.

Application for postponed delivery

If you need to apply for a postponement of the submission deadline due to illness or other reasons, you have to contact the Student Administration at the Department of Mathematics (e-mail: studieinfo@math.uio.no) well before the deadline.

All mandatory assignments in this course must be approved in the same semester, before you are allowed to take the final examination.

Complete guidelines about delivery of mandatory assignments:

uio.no/english/studies/admin/compulsory-activities/mn-math-mandatory.html

Definitions

Everywhere in this problem set, $\Omega \subset \mathbb{R}^n$ is an open, bounded domain with C^1 boundary.

Definitions.

1. Let H be a Hilbert space. A function $B: H \times H \rightarrow \mathbb{R}$ is a *bilinear form* if B is linear in each component.
2. If B is a bilinear form on a Hilbert space H , then it is
 - *continuous* if there exists $\alpha > 0$ such that $|B[u, v]| \leq \alpha \|u\|_H \|v\|_H$ for all $u, v \in H$
 - *coercive* if there exists $\beta > 0$ such that $B[u, u] \geq \beta \|u\|_H^2$ for all $u \in H$
 - *symmetric* if $B[u, v] = B[v, u]$ for all $u, v \in H$.
3. For any bounded set $\Omega \subset \mathbb{R}^n$ we write $|\Omega| := \int_{\Omega} 1 \, dx$, the volume of Ω .
4. If V is a vector space then a function $J: V \rightarrow \mathbb{R}$ is *strictly convex* if for all $u, v \in V$ with $u \neq v$ and all $s \in (0, 1)$,

$$J[su + (1-s)v] < sJ[u] + (1-s)J[v].$$

Problems

1. Consider the following version of Poincaré's inequality:

Theorem. Let $p \in [1, \infty]$ and let $\Omega \subset \mathbb{R}^n$ be open, bounded, connected and with C^1 boundary. For $u \in W^{1,p}(\Omega)$, define $\bar{u} := \frac{1}{|\Omega|} \int_{\Omega} u(x) \, dx$. Then there is some $C > 0$ such that

$$\|u - \bar{u}\|_{L^p(\Omega)} \leq C \|Du\|_{L^p(\Omega)} \quad \text{for all } u \in W^{1,p}(\Omega). \quad (1)$$

In this problem we will let $n = 1$ and $p < \infty$, so that Ω is a bounded interval, $\Omega := (a, b)$ for $a < b$.

- (a) Let $u \in C^1(\Omega) \cap C(\overline{\Omega})$ and let $v := u - \bar{u}$. Show that there is some $\bar{x} \in \Omega$ where $v(\bar{x}) = 0$.
- (b) Use the fundamental theorem of calculus to show that

$$|v(x)| \leq \int_a^b |v'(y)| \, dy \quad \forall x \in (a, b).$$

Use this and Hölder's inequality to prove (1) for all smooth u .

- (c) Prove (1) for all $u \in W^{1,p}(\Omega)$.

Solution:

- (a) The function v satisfies $\int_a^b v \, dx = 0$, so either $v \equiv 0$, or it attains both positive and negative values; since it is continuous, it must also attain the value 0.

(b) Let $\bar{x}$ be a zero of v . Then for any $x \in \Omega$,

$$|v(x)| = \left| v(\bar{x}) + \int_{\bar{x}}^x v'(y) dy \right| \leq \int_{\bar{x}}^x |v'(y)| dy \leq \int_a^b |v'(y)| dy.$$

By Hölder's inequality,

$$\int_a^b |v'(y)| dy \leq \|v'\|_{L^p(\Omega)} \|1\|_{L^q(\Omega)} = (b-a)^{1-1/p} \|v'\|_{L^p(\Omega)}$$

where q is the conjugate exponent to p . Hence,

$$\|v\|_{L^p(\Omega)} \leq (b-a)^{1-1/p} \|v'\|_{L^p(\Omega)} \|1\|_{L^p(\Omega)} = (b-a) \|v'\|_{L^p(\Omega)}.$$

(c) For every $u \in W^{1,p}(\Omega)$ we know there are $\varphi_1, \varphi_2, \dots \in C_c^\infty(\mathbb{R})$ so that $\varphi_k \rightarrow u$ as $k \rightarrow \infty$ in $W^{1,p}(\Omega)$. In particular, $\varphi_k - \varphi_{\bar{k}} \rightarrow u - \bar{u}$ in L^p and $D\varphi_k \rightarrow Du$ in L^p . Since (1) applies to all φ_k , we can let $k \rightarrow \infty$ and get the same conclusion for u .

2. Consider the boundary value problem

$$\begin{cases} -\Delta u + u = f & \text{in } \Omega \\ \alpha u + \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \quad (2)$$

for some $f \in L^2(\Omega)$ and $\alpha > 0$. (Here, $\nu(x)$ is the outward pointing unit normal vector at $x \in \partial\Omega$, and $\frac{\partial u}{\partial \nu}(x) = Du(x) \cdot \nu(x)$ is the normal derivative of u on $\partial\Omega$.)

(a) Show that any classical solution of (2) also satisfies $B[u, v] = (f, v)_{L^2}$ for any $v \in C^\infty(\mathbb{R}^d)$, where

$$B[u, v] := \int_{\Omega} Du \cdot Dv + uv \, dx + \alpha \int_{\partial\Omega} uv \, dS(x). \quad (3)$$

(b) Explain why $B[u, v]$ is still well-defined for $u, v \in H^1(\Omega)$.

(c) Show that B is a continuous and coercive bilinear form on $H^1(\Omega)$.

(d) Show that (2) has a unique weak solution $u \in H^1(\Omega)$ for any $f \in L^2(\Omega)$.

Solution:

(a) Multiply the equation by $v \in C^\infty(\mathbb{R}^d)$, integrate over Ω and apply integration by parts:

$$\begin{aligned} \int_{\Omega} f v \, dx &= - \int_{\Omega} v \Delta u \, dx + \int_{\Omega} +cuv \, dx \\ &= - \int_{\partial\Omega} v Du \cdot \nu \, dS(x) + \int_{\Omega} Du \cdot Dv \, dx + \int_{\Omega} cuv \, dx \\ &= \alpha \int_{\partial\Omega} v u \cdot \nu \, dS(x) + \int_{\Omega} Du \cdot Dv \, dx + \int_{\Omega} cuv \, dx, \end{aligned}$$

where we in the last step applied the boundary condition. This is precisely

$$B[u, v] = (f, v)_{L^2}.$$

- (b) If $u, v \in H^1(\Omega)$ and Du, Dv are the weak gradients of u, v , then u, v, Du and Dv all lie in $L^2(\Omega)$, so by Hölder's inequality, the functions $Du \cdot Dv$ and uv are integrable. By the trace theorem, there is a continuous linear operator $T: H^1(\Omega) \rightarrow L^2(\partial\Omega)$ so that " $Tu = u|_{\partial\Omega}$ ". Identifying the functions u, v in the boundary integral in (3) with their traces, we see that the integrand is both well-defined and integrable — again following from Hölder's inequality and the fact that $Tu, Tv \in L^2(\partial\Omega)$.

- (c) B is clearly bilinear. It is continuous since, by Hölder's inequality,

$$\begin{aligned} B[u, v] &\leq \|Du\|_{L^2(\Omega)} \|Dv\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \\ &\quad + \alpha \|Tu\|_{L^2(\partial\Omega)} \|Tv\|_{L^2(\partial\Omega)} \\ &= \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)} + \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)} \\ &\quad + \alpha C^2 \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)} \\ &= \tilde{C} \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)} \end{aligned}$$

where C is the continuity constant of T and $\tilde{C} = \alpha C^2 + 2$. To see that B is coercive, we have

$$B[u, u] = \underbrace{\int_{\Omega} |Du|^2 + u^2 dx}_{= \|u\|_{H^1(\Omega)}^2} + \underbrace{\alpha \int_{\partial\Omega} u^2 dS(x)}_{\geq 0} \geq \|u\|_{H^1(\Omega)}^2$$

- (d) Let $g: H^1(\Omega) \rightarrow \mathbb{R}$ be the functional $g(v) := \int_{\Omega} f v dx$. Then g is linear and continuous. Since B is a continuous, coercive bilinear form, it follows from the Lax–Milgram theorem that there is a unique $u \in H$ such that $B[u, v] = g(v)$ for all $v \in H$ — in other words, u is a weak solution of (2).

3. Let $\Omega \subset \mathbb{R}^n$ be open, bounded, connected and with C^1 boundary. For $u, v \in H^1(\Omega)$, define

$$B[u, v] := \int_{\Omega} Du \cdot Dv dx + \left(\int_{\Omega} u dx \right) \left(\int_{\Omega} v dx \right). \quad (4)$$

- (a) Show that B is a continuous, coercive bilinear form on $H^1(\Omega)$.
(b) Show that the weak formulation

$$B[u, v] = (f, v)_{L^2} \text{ for all } v \in H^1(\Omega) \quad (5)$$

has a unique solution $u \in H^1(\Omega)$ for any $f \in L^2(\Omega)$.

- (c) Show that the solution u satisfies $\int_{\Omega} u dx = \bar{f} := \frac{1}{|\Omega|} \int_{\Omega} f dx$.
(d) What is the problem of which u is a weak solution?
(e) What can we say about the relation between two solutions u_1, u_2 when the corresponding right-hand sides f_1, f_2 only differ by a constant?

Solution:

(a) B is clearly bilinear. It is continuous since, by Hölder's inequality,

$$\begin{aligned}
 B[u, v] &\leq \|Du\|_{L^2(\Omega)} \|Dv\|_{L^2(\Omega)} + \left(\int_{\Omega} |u| dx \right) \left(\int_{\Omega} |v| dx \right) \\
 &\leq \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)} \\
 &\quad + \|u\|_{L^2(\Omega)} \left(\int_{\Omega} 1^2 dx \right)^{1/2} \|v\|_{L^2(\Omega)} \left(\int_{\Omega} 1^2 dx \right)^{1/2} \\
 &= \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)} + |\Omega| \|u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \\
 &\leq C \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)}
 \end{aligned}$$

where $C := 1 + |\Omega|$.

Last, we claim that B is coercive. Let $\bar{u} := \frac{1}{|\Omega|} \int_{\Omega} u dx$. We have

$$\begin{aligned}
 \|u - \bar{u}\|_{L^2}^2 &= \int_{\Omega} u(x)^2 - 2\bar{u}u(x) + \bar{u}^2 dx \\
 &= \int_{\Omega} u(x)^2 dx - |\Omega| \bar{u}^2 = |\Omega| (\|u\|_{L^2}^2 - \bar{u}^2).
 \end{aligned}$$

By Poincaré's inequality we know that the above is bounded by $C \|Du\|_{L^2}^2$; reorganizing this inequality, we get

$$\bar{u}^2 \geq \|u\|_{L^2}^2 - \frac{C}{|\Omega|} \|Du\|_{L^2}^2.$$

Let $c \in (0, 1)$, to be determined. Then

$$\begin{aligned}
 B[u, u] &= \int_{\Omega} |Du|^2 dx + \left(\int_{\Omega} u dx \right)^2 = \|Du\|_{L^2}^2 + \bar{u}^2 \\
 &= \|Du\|_{L^2}^2 + (1 - c) \bar{u}^2 + c \bar{u}^2 \\
 &\geq \|Du\|_{L^2}^2 + (1 - c) \bar{u}^2 + c \left(\|u\|_{L^2}^2 - \frac{C}{|\Omega|} \|Du\|_{L^2}^2 \right) \\
 &\geq \|Du\|_{L^2}^2 \left(1 - c \frac{C}{|\Omega|} \right) + c \|u\|_{L^2}^2.
 \end{aligned}$$

Now pick $c > 0$ small enough that $1 - c \frac{C}{|\Omega|} > 0$. Setting $\beta := \min(c, 1 - c \frac{C}{|\Omega|})$, we get

$$B[u, u] \geq \beta \|u\|_{H^1}^2,$$

so B is coercive.

(b) This follows from the Lax–Milgram theorem.

(c) Let $v \equiv 1$, so that $Dv \equiv 0$. Inserting v in (5) gives

$$\begin{aligned} B[u, v] &= \int_{\Omega} Du \cdot Dv \, dx + \left(\int_{\Omega} u \, dx \right) \left(\int_{\Omega} 1 \, dx \right) \\ &= |\Omega| \bar{u} = (f, v) = \int_{\Omega} f(x) \, dx = |\Omega| \bar{f}. \end{aligned}$$

Dividing by $|\Omega|$ yields the claim.

(d) Assume that u is a smooth function satisfying (5). Then for any smooth function v ,

$$\begin{aligned} 0 &= B[u, v] - (f, v)_{L^2} = \int_{\Omega} Du \cdot Dv - f v \, dx + |\Omega| \bar{f} \int_{\Omega} v \, dx \\ &= \int_{\Omega} Du \cdot Dv - f v + \bar{f} v \, dx \\ &= \int_{\partial\Omega} v Du \cdot \nu \, dS(x) + \int_{\Omega} (-\Delta u - f + \bar{f}) v \, dx. \end{aligned}$$

Picking v with compact support in Ω yields

$$\int_{\Omega} (-\Delta u - f + \bar{f}) v \, dx = 0 \quad \forall v \in C_c^\infty,$$

so necessarily,

$$-\Delta u = f - \bar{f} \quad \text{in } \Omega.$$

On the other hand, picking functions v which vanish everywhere except close to $\partial\Omega$, we find that the $\int_{\Omega}$ integral vanishes, so the first integral must be 0 for any such v , whence

$$\frac{\partial u}{\partial \nu} := Du \cdot \nu = 0.$$

(This is a *Neumann boundary condition*.)

(e) Assume that $B[u_j, v] = (f_j, v)_{L^2}$ for $j = 1, 2$ for all $v \in H^1(\Omega)$, where $f_2(x) = f_1(x) + c$ for all $x \in \Omega$ for some $c \in \mathbb{R}$. Formally speaking, u_1, u_2 satisfy the same PDE (since $f_1 - \bar{f}_1 = f_2 - \bar{f}_2$), but have different average, so we might suspect that also u_1, u_2 differ by a constant. Let $w(x) := u_1(x) + b$ for $x \in \Omega$, for some $b \in \mathbb{R}$. Then $w \in H^1(\Omega)$, and if $v \in \Omega$ then

$$\begin{aligned} B[w, v] &= \int_{\Omega} Du_1 \cdot Dv \, dx + \left(\int_{\Omega} u_1 + b \, dx \right) \left(\int_{\Omega} v \, dx \right) \\ &= B[u_1, v] + b|\Omega| \int_{\Omega} v \, dx \\ &= \int_{\Omega} f_1 v \, dx + b|\Omega| \int_{\Omega} v \, dx \\ &= \int_{\Omega} (f_1 + b|\Omega|) v \, dx. \end{aligned}$$

If $b = \frac{c}{|\Omega|}$ then $f_1 + b|\Omega| = f_2$, and therefore $B[w, v] = (f_2, v)_{L^2}$ for all v . But u_2 is the unique solution of this problem, so $w = u_2$ — in other words,

$$u_2(x) = u_1(x) + \frac{c}{|\Omega|} \quad \forall x \in \Omega.$$

4. Let $B: H \times H \rightarrow H$ be a symmetric, continuous and coercive bilinear form on a Hilbert space H . Fix some $f \in H$ and define the “energy functional” $J: H \rightarrow \mathbb{R}$ by

$$J[v] := B[v, v] - 2(f, v)_H, \quad v \in H.$$

- (a) Show that J is strictly convex.
- (b) Show that $J[v] \geq -\frac{\|f\|^2}{\beta}$ for all $v \in H$ (where β is the coercivity constant of B) and that $J[v] \rightarrow \infty$ as $\|v\| \rightarrow \infty$.
- (c) Show that J has at most one minimizer, that is, there is at most one $u \in H$ such that $J[u] = \inf_{v \in H} J[v]$.
- (d) Show that if $u \in H$ minimizes J , then u solves

$$B[u, v] = (f, v)_H \text{ for all } v \in H. \quad (6)$$

- (e) Conversely, show that if $u \in H$ solves (6) then u minimizes J .

Remark. This problem shows that the solution of an elliptic PDE is precisely that function which minimizes the associated *energy*. Thus, there is a close connection between PDEs (in particular, their weak formulation (6)) and the *calculus of variations* — the mathematical theory of minimizing functions. For this reason, the weak formulation (6) is sometimes called the *variational formulation* of the PDE.

Solution:

- (a) Let $u, v \in H$ and let $s \in (0, 1)$. Then

$$\begin{aligned} J[su + (1-s)v] &= sJ[u] + (1-s)J[v] \\ &= B[su + (1-s)v, su + (1-s)v] - sB[u, u] - (1-s)B[v, v] \\ &\quad - 2(f, su + (1-s)v) + 2s(f, u) + 2(1-s)(f, v) \\ &= (s^2 - s)B[u, u] + s(1-s)B[u, v] + s(1-s)B[v, u] + ((1-s)^2 - (1-s))B[v, v] \\ &= -s(1-s)(B[u, u] - B[u, v] - B[v, u] + B[v, v]) \\ &= -s(1-s)B[u - v, u - v] \\ &\leq -\beta s(1-s)\|u - v\|^2 \end{aligned}$$

where $\beta > 0$ is the coercivity constant of B . Hence, if $s \in (0, 1)$ and $u \neq v$, then the above is strictly negative. This proves strict convexity.

- (b) By coercivity and Hölder’s inequality,

$$J[v] \geq \beta\|v\|^2 - 2\|f\|\|v\|.$$

The minimum of the quadratic function $s \mapsto \beta s^2 - 2\|f\|s$ is attained at

$s = \frac{\|f\|}{\beta}$, and the minimum is $-\frac{\|f\|^2}{\beta}$. This proves the lower bound. From the above estimate it is also clear that $J[v] \rightarrow \infty$ as $\|v\| \rightarrow \infty$.

- (c) Assume both u_1 and u_2 minimize J , that is, $J[u_1] = J[u_2] = \inf_{v \in H} J[v]$, but that $u_1 \neq u_2$. If $w := \frac{u_1 + u_2}{2}$, then by strict convexity,

$$J[w] = J[\frac{1}{2}u_1 + \frac{1}{2}u_2] < \frac{1}{2}J[u_1] + \frac{1}{2}J[u_2] = \inf_{v \in V} J[v] \leq J[w],$$

a contradiction.

- (d) Let u be a minimizer, pick any $v \in H$, and let $\varepsilon > 0$. Then

$$\begin{aligned} 0 &\leq J[u + \varepsilon v] - J[u] \\ &= B[u + \varepsilon v, u + \varepsilon v] - B[u, u] - 2(f, u + \varepsilon v) + 2(f, u) \\ &= \varepsilon(B[u, v] + B[v, u]) + \varepsilon^2 B[v, v] - 2\varepsilon(f, v) \\ &= 2\varepsilon(B[u, v] - (f, v)) + \varepsilon^2 B[v, v], \end{aligned} \tag{7}$$

where we in the last step used the symmetry of B . Hence, dividing the above by 2ε and passing $\varepsilon \rightarrow 0$ gives

$$0 \leq B[u, v] - (f, v) \text{ for all } v \in H.$$

Replacing v by $-v$ in (7) gives

$$0 \leq -2\varepsilon(B[u, v] - (f, v)) + \varepsilon^2 B[v, v].$$

Again dividing by 2ε and letting $\varepsilon \rightarrow 0$ gives the opposite inequality

$$0 \geq B[u, v] - (f, v) \text{ for all } v \in H.$$

Together, these yield (6).

- (e) Let u solve (6) and let $v \in H$. By the same computation as in the previous problem, we have

$$J[u + v] - J[u] = 2(B[u, v] - (f, v)) + B[v, v] = B[v, v] \geq \beta\|v\|^2.$$

Hence, if $w \in H$ does not equal u and $v := w - u$, then

$$J[w] - J[u] \geq \beta\|u - w\|^2 > 0,$$

in other words, u is a strict minimum of J .

Hints

2. (b) You need to use the trace operator to make sense of u, v on $\partial\Omega$. Make sure that you show that $Du \cdot Dv$ and uv are integrable on Ω , and that uv is integrable on $\partial\Omega$. (Integrable means that the absolute value has finite integral.)
3. (a) For coercivity you need to use Poincaré's inequality with $p = 2$. Start by writing out $\|u - \bar{u}\|_{L^2}^2$.
(c) Let v be a constant in the weak formulation.
4. (c) Assume that J has two minimizers u, v . Use the strict convexity of J .
(d) Let u be a minimizer. Write out $J[u + \varepsilon v] - J[u]$ for arbitrary $\varepsilon > 0$ and $v \in H$.
(e) Let u solve (6). Write out $J[u + v] - J[u]$ for an arbitrary $v \in H$.