

Problem sheet 1

MAT4305

Ulrik Skre Fjordholm

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1. The *standard mollifier* or *Friedrichs mollifier* is the function

$$\omega = c_n \tilde{\omega}, \quad \tilde{\omega}(x) := \begin{cases} \exp((|x|^2 - 1)^{-1}) & \text{for } |x| < 1 \\ 0 & \text{for } |x| \geq 1 \end{cases}$$

where $c_n > 0$ is the constant that ensures that $\int_{\mathbb{R}^n} \omega(x) dx = 1$. We claim that $\omega \in C_c^\infty(\mathbb{R}^n)$.

- (a) Let $\alpha \in \mathbb{N}_0^n$ be a multiindex. Show that the α -th partial derivative of ω can be written as

$$D^\alpha \omega(x) = \frac{P_\alpha(x)}{(|x|^2 - 1)^{2|\alpha|}} \exp((|x|^2 - 1)^{-1}) \quad \text{for } |x| < 1$$

for some polynomial $P_\alpha: \mathbb{R}^n \rightarrow \mathbb{R}$. (You do not have to find P_α .)

Hint. Use induction on $|\alpha|$. It might be easiest to let $n = 1$.

- (b) Use (a) to prove that

$$\lim_{\substack{|x| \rightarrow 1 \\ |x| < 1}} D^\alpha \omega(x) = 0$$

for any multiindex α .

- (c) Conclude from (b) that $\omega \in C_c^\infty(\mathbb{R}^n)$.

2. The *indicator function* of a set $\Omega \subset \mathbb{R}^n$ is the function

$$\mathbb{1}_\Omega(x) := \begin{cases} 1 & \text{for } x \in \Omega \\ 0 & \text{for } x \notin \Omega. \end{cases}$$

This function is clearly discontinuous whenever Ω is nonempty. For $\varepsilon > 0$, let $\chi_\varepsilon := \mathbb{1}_\Omega * \omega_\varepsilon$, the mollification of $\mathbb{1}_\Omega$.

- (a) Write out χ_ε as an integral of ω_ε over Ω .
(b) Show directly that $\chi_\varepsilon \in C^\infty(\mathbb{R}^n)$. (You may use the fact that $\omega_\varepsilon \in C_c^\infty(\mathbb{R}^n)$.)
(c) The support of $\mathbb{1}_\Omega$ is $\overline{\Omega}$. Can you find the support of χ_ε ?

Hint. Recall that $\text{supp } f := \text{closure}(\{x : f(x) \neq 0\})$.

- (d) Show that $0 \leq \chi_\varepsilon(x) \leq 1$ for all $x \in \mathbb{R}^n$.
(e) Show directly that $\chi_\varepsilon(x) \rightarrow 1$ as $\varepsilon \rightarrow 0$ for every $x \in \text{int}(\Omega)$, and $\chi_\varepsilon \rightarrow 0$ for every $x \in \text{int}(\Omega^c)$.

Hint. The set $\text{int}(\Omega)$ is the internal of Ω , consisting of all $x \in \Omega$ for which there is some $\delta > 0$ such that $B(x, \delta) \subset \Omega$.

3. Compute the following distributional derivatives:

$$D^{(1,0,2)}u, \quad \text{where } u(x) = (x_1)^3 \sin(x_2 + 3x_3) \quad \text{for } x \in \mathbb{R}^3.$$

$$\frac{d}{dx}v, \quad \text{where } v(x) = \begin{cases} 0 & x < 0 \\ x^2 & 0 < x < 1 \\ 2 & x > 1 \end{cases} \quad \text{for } x \in \mathbb{R}.$$

$$\frac{\partial}{\partial x_1}w, \quad \text{where } w(x) = \begin{cases} (x_2)^2 & x_1 < 0 \\ 1 - (x_2)^2 & x_1 > 0 \end{cases} \quad \text{for } x \in \mathbb{R}^2.$$

$$\frac{\partial^2}{\partial x_1 \partial x_2}\delta, \quad \text{where } \delta \text{ is the Dirac distribution, } \delta(\varphi) = \varphi(0) \text{ for } \varphi \in \mathcal{D}(\mathbb{R}^2).$$

$$\frac{d}{dx}c, \quad \text{where } c \text{ is the Cantor function on } [0, 1].$$

Hint. Here and elsewhere, we identify a function with the corresponding distribution. Thus, when we write e.g. $\frac{d}{dx}v$, we mean the (first) distributional derivative of the distribution T_v defined by $T_v(\varphi) = \int v(x)\varphi(x) dx$.

4. Let $\Omega = B^0(0, 1) \subset \mathbb{R}^n$, the open unit ball in $\mathbb{R}^n$, and define $u: \Omega \rightarrow \mathbb{R}$ by $u(x) = |x|^\gamma$ for some $\gamma \in \mathbb{R}$. For which $k \in \mathbb{N}_0$ and $\beta \in [0, 1]$ does u lie in the Hölder space $C^{k,\beta}(\Omega)$?

Hint. Start with $\gamma \leq 0$, then move on to $\gamma \in (0, 1]$.

5. For a distribution $T \in \mathcal{D}'(\Omega)$ and a multiindex $\alpha \in \mathbb{N}_0^n$ we define the α -th distributional derivative of T as $D^\alpha T$, defined by

$$D^\alpha T(\varphi) := (-1)^{|\alpha|} T(D^\alpha \varphi), \quad \varphi \in \mathcal{D}(\Omega).$$

Show that $D^\alpha T$ is a distribution.

Hint. Recall that $S := D^\alpha T$ is a distribution if and only if (i) S is a well-defined map from $\mathcal{D}(\Omega)$ to $\mathbb{R}$, (ii) S is linear, and (iii) S is continuous, i.e. if $\varphi_k \rightarrow \varphi$ as $k \rightarrow \infty$ then also $S(\varphi_k) \rightarrow S(\varphi)$. Recall also that when we write $\varphi_k \rightarrow \varphi$ we mean that $\|\varphi_k - \varphi\|_{C(\Omega)} \rightarrow 0$ and $\|D^\beta \varphi_k - D^\beta \varphi\|_{C(\Omega)} \rightarrow 0$ as $k \rightarrow \infty$ for all multiindices β .