

Problem sheet 1 – Solutions

MAT4305

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1. The *standard mollifier* or *Friedrichs mollifier* is the function

$$\omega = c_n \tilde{\omega}, \quad \tilde{\omega}(x) := \begin{cases} \exp((|x|^2 - 1)^{-1}) & \text{for } |x| < 1 \\ 0 & \text{for } |x| \geq 1 \end{cases}$$

where $c_n > 0$ is the constant that ensures that $\int_{\mathbb{R}^n} \omega(x) dx = 1$. We claim that $\omega \in C_c^\infty(\mathbb{R}^n)$.

(a) Let $\alpha \in \mathbb{N}_0^n$ be a multiindex. Show that the α -th partial derivative of ω can be written as

$$D^\alpha \omega(x) = \frac{P_\alpha(x)}{(|x|^2 - 1)^{2|\alpha|}} \exp((|x|^2 - 1)^{-1}) \quad \text{for } |x| < 1$$

for some polynomial $P_\alpha: \mathbb{R}^n \rightarrow \mathbb{R}$. (You do not have to find P_α .)

Hint. Use induction on $|\alpha|$. It might be easiest to let $n = 1$.

(b) Use (a) to prove that

$$\lim_{\substack{|x| \rightarrow 1 \\ |x| < 1}} D^\alpha \omega(x) = 0$$

for any multiindex α .

(c) Conclude from (b) that $\omega \in C_c^\infty(\mathbb{R}^n)$.

Solution:

(a) We do this by induction. It is clearly true for $\alpha = 0$. Assume that it has been proven for all multiindices of length $|\alpha| = k$. If α has length $|\alpha| = k + 1$ then it can be written as $\alpha = \tilde{\alpha} + e_i$ for some $i \in \{1, \dots, n\}$ and multiindex of length $|\tilde{\alpha}| = k$. Then

$$\begin{aligned} D^\alpha \omega(x) &= D^{e_i} D^{\tilde{\alpha}} \omega(x) = \frac{\partial}{\partial x_i} \left(\frac{P_{\tilde{\alpha}}(x)}{(|x|^2 - 1)^{2|\tilde{\alpha}|}} \exp((|x|^2 - 1)^{-1}) \right) \\ &= \left(\frac{\partial}{\partial x_i} \left(\frac{P_{\tilde{\alpha}}(x)}{(|x|^2 - 1)^{2|\tilde{\alpha}|}} \right) + \frac{P_{\tilde{\alpha}}(x)}{(|x|^2 - 1)^{2|\tilde{\alpha}|}} \frac{-2x_i}{(|x|^2 - 1)^2} \right) \exp((|x|^2 - 1)^{-1}) \\ &= \frac{P_\alpha(x)}{(|x|^2 - 1)^{2|\alpha|+2}} \exp((|x|^2 - 1)^{-1}) \end{aligned}$$

for some polynomial P_α .

(b) If $x \rightarrow x_0 \in \partial B(0, 1)$ then

$$\lim_{x \rightarrow x_0} D^\alpha \omega(x) = P_\alpha(x_0) \lim_{x \rightarrow x_0} \frac{\exp((|x|^2 - 1)^{-1})}{(|x|^2 - 1)^{2|\alpha|}} = P_\alpha(x_0) \lim_{s \rightarrow 0} \frac{\exp(s^{-1})}{s^{2|\alpha|}} = 0$$

Since $e^{1/s} \rightarrow 0$ as $s \rightarrow 0$ faster than any polynomial.

(c) We have now established that $D^\alpha \omega$ is identically equal to 0 outside of $B(0, 1)$, while inside of $B(0, 1)$ it is equal to a continuous function which goes to 0 as we approach $\partial B(0, 1)$. It follows that $D^\alpha \omega$ is continuous for any α , whence $\omega \in C^\infty$. Since it also has compact support, we conclude $\omega \in C_c^\infty(\mathbb{R}^n)$.

2. The *indicator function* of a set $\Omega \subset \mathbb{R}^n$ is the function

$$\mathbb{1}_\Omega(x) := \begin{cases} 1 & \text{for } x \in \Omega \\ 0 & \text{for } x \notin \Omega. \end{cases}$$

This function is clearly discontinuous whenever Ω is nonempty. For $\varepsilon > 0$, let $\chi_\varepsilon := \mathbb{1}_\Omega * \omega_\varepsilon$, the mollification of $\mathbb{1}_\Omega$.

(a) Write out χ_ε as an integral of ω_ε over Ω .

(b) Show directly that $\chi_\varepsilon \in C^\infty(\mathbb{R}^n)$. (You may use the fact that $\omega_\varepsilon \in C_c^\infty(\mathbb{R}^n)$.)

(c) The support of $\mathbb{1}_\Omega$ is $\overline{\Omega}$. Can you find the support of χ_ε ?

Hint. Recall that $\text{supp } f := \text{closure}(\{x : f(x) \neq 0\})$.

(d) Show that $0 \leq \chi_\varepsilon(x) \leq 1$ for all $x \in \mathbb{R}^n$.

(e) Show directly that $\chi_\varepsilon(x) \rightarrow 1$ as $\varepsilon \rightarrow 0$ for every $x \in \text{int}(\Omega)$, and $\chi_\varepsilon \rightarrow 0$ for every $x \in \text{int}(\Omega^c)$.

Hint. The set $\text{int}(\Omega)$ is the internal of Ω , consisting of all $x \in \Omega$ for which there is some $\delta > 0$ such that $B(x, \delta) \subset \Omega$.

Solution:

(a) We can write

$$\chi_\varepsilon(x) = \int_{\mathbb{R}^n} \mathbb{1}_\Omega(x - y) \omega_\varepsilon(y) dy = \int_{\mathbb{R}^n} \mathbb{1}_\Omega(z) \omega_\varepsilon(x - z) dz = \int_{\Omega} \omega_\varepsilon(x - z) dz.$$

(b) Since ω_ε and all of its derivatives lie in C_c^∞ , we can differentiate under the integral sign to get

$$D^\alpha \chi_\varepsilon(x) = \int_{\Omega} D^\alpha \omega_\varepsilon(x - z) dz,$$

which is continuous, since $D^\alpha \omega_\varepsilon$ is.

(c) First, note that $\chi_\varepsilon(x) \geq 0$ for all x . We have

$$\chi_\varepsilon(x) = \int_{\Omega} \omega_\varepsilon(x - z) dz = \int_{\Omega \cap B(x, \varepsilon)} \omega_\varepsilon(x - z) dz$$

(since $\text{supp } \omega_\varepsilon(x - \cdot) = B(x, \varepsilon)$). In order for $\chi_\varepsilon(x) > 0$ we would need the volume $|\Omega \cap B(x, \varepsilon)|$ to be positive. Identifying such points would lead us too far astray

into measure theory, but certainly, if $\text{dist}(x, \Omega) \geq \varepsilon$ then the above volume must be zero. Hence,

$$\text{supp } \chi_\varepsilon \subset \Omega_\varepsilon := \{x \in \mathbb{R}^n : \text{dist}(x, \Omega) \leq \varepsilon\}.$$

(It can be shown that if Ω is open, or is the closure of an open set, then there is equality in the above inclusion.)

- (d) χ_ε is the integral of a nonnegative function, and hence is nonnegative. On the other hand,

$$\chi_\varepsilon(x) = \int_{\Omega} \omega_\varepsilon(x-z) dz \leq \int_{\mathbb{R}^n} \omega_\varepsilon(x-z) dz = 1.$$

- (e) Let $x \in \text{int}(\Omega)$, and let $\delta > 0$ be such that $B(x, \delta) \subset \Omega$. If $\varepsilon \leq \delta$ then $\text{supp } \omega_\varepsilon(x - \cdot) = B(x, \varepsilon) \subset B(x, \delta) \subset \Omega$, so

$$\chi_\varepsilon(x) = \int_{\Omega} \omega_\varepsilon(x-z) dz = \int_{\mathbb{R}^n} \omega_\varepsilon(x-z) dz = 1.$$

Hence, $\chi_\varepsilon(x) \rightarrow 1$ as $\varepsilon \rightarrow 0$. An analogous argument works for $x \in \text{int}(\Omega^c)$.

3. Compute the following distributional derivatives:

$$D^{(1,0,2)}u, \quad \text{where } u(x) = (x_1)^3 \sin(x_2 + 3x_3) \quad \text{for } x \in \mathbb{R}^3.$$

$$\frac{d}{dx}v, \quad \text{where } v(x) = \begin{cases} 0 & x < 0 \\ x^2 & 0 < x < 1 \\ 2 & x > 1 \end{cases} \quad \text{for } x \in \mathbb{R}.$$

$$\frac{\partial}{\partial x_1}w, \quad \text{where } w(x) = \begin{cases} (x_2)^2 & x_1 < 0 \\ 1 - (x_2)^2 & x_1 > 0 \end{cases} \quad \text{for } x \in \mathbb{R}^2.$$

$$\frac{\partial^2}{\partial x_1 \partial x_2} \delta, \quad \text{where } \delta \text{ is the Dirac distribution, } \delta(\varphi) = \varphi(0) \text{ for } \varphi \in \mathcal{D}(\mathbb{R}^2).$$

$$\frac{d}{dx}c, \quad \text{where } c \text{ is the Cantor function on } [0, 1].$$

Hint. Here and elsewhere, we identify a function with the corresponding distribution. Thus, when we write e.g. $\frac{d}{dx}v$, we mean the (first) distributional derivative of the distribution T_v defined by $T_v(\varphi) = \int v(x)\varphi(x) dx$.

Solution: u is a smooth function, so its distributional derivatives coincide with the classical derivatives. We get

$$D^{(1,0,2)}u(x) = \frac{\partial^3 u}{\partial x_1 \partial x_2^2}(x) = -27(x_1)^2 \sin(x_2 + 3x_3).$$

The function v is differentiable apart from at $x = 0, 1, 2$, so away from these points we get $\frac{dv}{dx}(x) = 0$ for $x < 0$ and $x > 1$, and $\frac{dv}{dx}(x) = 2x$ for $x \in (0, 1)$. The function has a jump discontinuity at $x = 1$, so the derivative there is a Dirac. In conclusion,

$$\frac{dv}{dx} = T_g + \delta_1, \quad \text{where } g(x) = \begin{cases} 0 & x < 0 \text{ or } x > 1, \\ 2x & 0 < x < 1 \end{cases}$$

and δ_1 is the Dirac distribution at $x = 1$, mapping test functions φ to $\varphi(1)$.

Let $\varphi \in C_c^\infty(\mathbb{R}^2)$. Writing $\mathbb{R}_+^2 = \{x \in \mathbb{R}^2 : x_1 > 0\}$ and $\mathbb{R}_-^2 = \mathbb{R}^2 \setminus \mathbb{R}_+^2$, we get

$$\int_{\mathbb{R}^2} w(x) \frac{\partial \varphi}{\partial x_1}(x) dx = \int_{\mathbb{R}_+^2} (1 - (x_2)^2) \frac{\partial \varphi}{\partial x_1}(x) dx + \int_{\mathbb{R}_-^2} (x_2)^2 \frac{\partial \varphi}{\partial x_1}(x) dx$$

(integration by parts; $\nu = (\nu_1, \nu_2) = (\mp 1, 0)$ is the outward pointing unit normal vector to $\mathbb{R}_\pm^2$)

$$\begin{aligned} &= \int_{\partial \mathbb{R}_+^2} (1 - (x_2)^2) \varphi(x) \underbrace{\nu_1(x)}_{=-1} dS(x) - \underbrace{\int_{\mathbb{R}_+^2} \frac{\partial}{\partial x_1} (1 - (x_2)^2) \varphi(x) dx}_{=0} \\ &\quad + \int_{\partial \mathbb{R}_-^2} (x_2)^2 \varphi(x) \underbrace{\nu_1(x)}_{=1} dS(x) - \underbrace{\int_{\mathbb{R}_-^2} \frac{\partial}{\partial x_1} (x_2)^2 \varphi(x) dx}_{=0} \\ &= \int_{\partial \mathbb{R}_+^2} ((x_2)^2 - 1) \varphi(x) dS(x) + \int_{\partial \mathbb{R}_-^2} (x_2)^2 \varphi(x) dS(x) \\ &= \int_{\mathbb{R}} \varphi(0, x_2) (2(x_2)^2 - 1) dx_2. \end{aligned}$$

Thus, $\frac{\partial w}{\partial x_1}$ is the distribution

$$\frac{\partial w}{\partial x_1}(\varphi) = \int_{\mathbb{R}} \varphi(0, x_2) (2(x_2)^2 - 1) dx_2.$$

(We can think of this as a “Dirac measure smeared along $\{x_1 = 0\}$, weighted by $2(x_2)^2 - 1$ ”.)

For the Dirac distribution we have

$$\frac{\partial^2 \delta}{\partial x_1 \partial x_2}(\varphi) = (-1)^2 \delta\left(\frac{\partial^2 \varphi}{\partial x_1 \partial x_2}\right) = \frac{\partial^2 \varphi}{\partial x_1 \partial x_2}(0).$$

For the Cantor function c , recall that this is a continuous, increasing function with $c(0) = 0$, $c(1) = 1$ and classical derivative $\frac{dc}{dx}(x) = 0$ at Lebesgue-almost every $x \in [0, 1]$ (in fact, in a dense, open subset of measure 1). Its distributional derivative is

$$\frac{dc}{dx}(\varphi) = -c\left(\frac{d\varphi}{dx}\right) = -\int_0^1 c(x) \frac{d\varphi}{dx}(x) dx.$$

The above *cannot* be written as $-T_g(\varphi) = -\int_0^1 g(x) \varphi(x) dx$ for some function g , since then g would have to coincide with $\frac{dc}{dx}(x) = 0$ in the dense, open subset where c is differentiable; this again would imply $g = 0$ a.e., leading to $T_g \equiv 0$. Since the distribution $\frac{dc}{dx}$ is not identically equal to zero (it can be shown that the derivative of a distribution is zero if and only if the distribution is constant, just as for C^1 functions), we conclude that the distributional derivative of c is not a function.

4. Let $\Omega = B^0(0, 1) \subset \mathbb{R}^n$, the open unit ball in $\mathbb{R}^n$, and define $u: \Omega \rightarrow \mathbb{R}$ by $u(x) = |x|^\gamma$ for some $\gamma \in \mathbb{R}$. For which $k \in \mathbb{N}_0$ and $\beta \in [0, 1]$ does u lie in the Hölder space $C^{k, \beta}(\Omega)$?

Hint. Start with $\gamma \leq 0$, then move on to $\gamma \in (0, 1]$.

Solution: If $\gamma \leq 0$ then u is discontinuous (with the convention $0^0 = 0$), so it doesn't lie in any Hölder space. If $\gamma \in (0, 1]$ then u is continuous but not differentiable (at the origin). If $f(t, h) := (t + h)^\gamma - t^\gamma$ for $t, h \geq 0$ then $\max_{t \geq 0} f(t, h) = f(0, h) = h^\gamma$. Hence, if $x, y \in \Omega$ and, say, $|x| \geq |y|$ then

$$|u(x) - u(y)| = |x|^\gamma - |y|^\gamma = f(|y|, |x| - |y|) \leq f(0, |x| - |y|) = (|x| - |y|)^\gamma.$$

The inverse triangle inequality states that $||x| - |y|| \leq |x - y|$, so

$$|u(x) - u(y)| \leq |x - y|^\gamma.$$

Therefore, $u \in C^{0,\gamma}(\Omega)$. We cannot hope for $u \in C^{0,\beta}(\Omega)$ for $\beta > \gamma$, since the above inequalities are sharp when $y = 0$.

For $\gamma > 1$, let $k \in \mathbb{N}$ be such that $k < \gamma \leq k + 1$. Then u is differentiable k times, with derivatives behaving like $|x|^{\gamma-1}$, $|x|^{\gamma-2}$, etc. It follows that $u \in C^{k,\gamma-k}(\Omega)$.

5. For a distribution $T \in \mathcal{D}'(\Omega)$ and a multiindex $\alpha \in \mathbb{N}_0^n$ we define the α -th distributional derivative of T as $D^\alpha T$, defined by

$$D^\alpha T(\varphi) := (-1)^{|\alpha|} T(D^\alpha \varphi), \quad \varphi \in \mathcal{D}(\Omega).$$

Show that $D^\alpha T$ is a distribution.

Hint. Recall that $S := D^\alpha T$ is a distribution if and only if (i) S is a well-defined map from $\mathcal{D}(\Omega)$ to $\mathbb{R}$, (ii) S is linear, and (iii) S is continuous, i.e. if $\varphi_k \rightarrow \varphi$ as $k \rightarrow \infty$ then also $S(\varphi_k) \rightarrow S(\varphi)$. Recall also that when we write $\varphi_k \rightarrow \varphi$ we mean that $\|\varphi_k - \varphi\|_{C(\Omega)} \rightarrow 0$ and $\|D^\beta \varphi_k - D^\beta \varphi\|_{C(\Omega)} \rightarrow 0$ as $k \rightarrow \infty$ for all multiindices β .

Solution: If $\varphi \in \mathcal{D}(\Omega)$ then so is $D^\alpha \varphi$ for any multiindex α ; it follows that “ $T(D^\alpha \varphi)$ ” makes sense, so $D^\alpha T$ is well-defined. $D^\alpha T$ is linear since both T and the act of differentiation is linear: If $\varphi, \psi \in \mathcal{D}(\Omega)$ and $\gamma, \lambda \in \mathbb{R}$ then

$$\begin{aligned} D^\alpha T(\gamma\varphi + \lambda\psi) &= (-1)^{|\alpha|} T(D^\alpha(\gamma\varphi + \lambda\psi)) = (-1)^{|\alpha|} T(\gamma D^\alpha \varphi + \lambda D^\alpha \psi) \\ &= \gamma(-1)^{|\alpha|} T(D^\alpha \varphi) + \lambda(-1)^{|\alpha|} T(D^\alpha \psi) = \gamma D^\alpha T(\varphi) + \lambda D^\alpha T(\psi). \end{aligned}$$

Last, we claim that T is continuous. Indeed, let $\{\varphi_m\}_{m \in \mathbb{N}}$ be a sequence in $\mathcal{D}(\Omega)$ converging to φ . Then $\{D^\alpha \varphi_m\}_{m \in \mathbb{N}}$ is also a sequence in $\mathcal{D}(\Omega)$ which converges to $D^\alpha \varphi$. It follows that

$$D^\alpha T(\varphi_m) = (-1)^{|\alpha|} T(D^\alpha \varphi_m) \rightarrow (-1)^{|\alpha|} T(D^\alpha \varphi) = D^\alpha T(\varphi).$$

This concludes the proof.