

Problem sheet 2 – Solutions

MAT4305

Ulrik Skre Fjordholm

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1. Prove parts (ii) and (iii) of Theorem 1 in §5.2.3: If $u, v \in W^{k,p}(\Omega)$ for an open $\Omega \subset \mathbb{R}^n$, $k \in \mathbb{N}$ and $p \in [1, \infty]$, then

(ii) $\lambda u + \mu v \in W^{k,p}(\Omega)$ for any $\lambda, \mu \in \mathbb{R}$, and

$$D^\alpha(\lambda u + \mu v) = \lambda D^\alpha u + \mu D^\alpha v$$

for every multiindex α with $|\alpha| \leq k$.

(iii) If $V \subset \Omega$ is open then the restriction $u|_V$ of u to V lies in $W^{k,p}(V)$

Solution: *Proof of (ii):* For any $\varphi \in C_c^\infty(\Omega)$ we have

$$\begin{aligned} \int_{\Omega} (\lambda u + \mu v) D^\alpha \varphi \, dx &= \lambda \int_{\Omega} u D^\alpha \varphi \, dx + \mu \int_{\Omega} v D^\alpha \varphi \, dx \\ &= (-1)^{|\alpha|} \lambda \int_{\Omega} D^\alpha u \varphi \, dx + (-1)^{|\alpha|} \mu \int_{\Omega} D^\alpha v \varphi \, dx \\ &= (-1)^{|\alpha|} \int_{\Omega} (\lambda D^\alpha u + \mu D^\alpha v) \varphi \, dx. \end{aligned}$$

This proves the claim.

Proof of (iii): We claim that $D^\alpha(u|_V) = (D^\alpha u)|_V$ for every α . First of all, $(D^\alpha u)|_V \in L^p(V)$ for every α since

$$\|(D^\alpha u)|_V\|_{L^p(V)} = \|D^\alpha u\|_{L^p(V)} \leq \|D^\alpha u\|_{L^p(\Omega)},$$

since we're integrating over a larger set. Next, observe that every $\varphi \in C_c^\infty(V)$ can be viewed as an element of $C_c^\infty(\Omega)$ by letting $\varphi|_{\Omega \setminus V} \equiv 0$. Therefore, for every $\varphi \in C_c^\infty(V)$,

$$\begin{aligned} \int_V u|_V D^\alpha \varphi \, dx &= \int_{\Omega} u D^\alpha \varphi \, dx && \text{(since } D^\alpha \varphi \equiv 0 \text{ outside of } V) \\ &= (-1)^{|\alpha|} \int_{\Omega} D^\alpha u \varphi \, dx && \text{(since } D^\alpha u \text{ is a weak derivative of } u) \\ &= (-1)^{|\alpha|} \int_V D^\alpha u \varphi \, dx && \text{(since } \varphi \equiv 0 \text{ outside of } V) \\ &= (-1)^{|\alpha|} \int_V (D^\alpha u)|_V \varphi \, dx. \end{aligned}$$

This proves our claim.

2. For which $k, n \in \mathbb{N}$ and $p \in [1, \infty]$ does each of the following functions lie in $W^{k,p}(\mathbb{R}^n)$?

$$\begin{aligned} u(x) &:= e^{-|x|^2} \\ v(x) &:= \min(1, |x|^\gamma) \quad \text{for some } \gamma < 0 \\ w(x) &:= \sin(|x|^2) \end{aligned}$$

Solution: The function u lies in C^∞ . Any derivative of u is of the form qu , where q is a polynomial. Since $qu \in L^p(\mathbb{R}^n)$ for any $p \in [1, \infty]$ for any polynomial q , we conclude that $u \in W^{k,p}(\mathbb{R}^n)$ for all choices of k, p, n .

We have

$$v(x) = \begin{cases} 1 & \text{if } |x| < 1 \\ |x|^{-\gamma} & \text{else} \end{cases} \Rightarrow \frac{\partial v}{\partial x_i}(x) = \begin{cases} 0 & \text{if } |x| < 1 \\ \gamma |x|^{\gamma-1} \frac{x_i}{|x|} & \text{if } |x| > 1. \end{cases}$$

One can now check that the a.e.-defined function $\frac{\partial v}{\partial x_i}$ is actually the i -th weak derivative of v . We have

$$\begin{aligned} \|v\|_{L^p(\mathbb{R}^n)} &= \left(\int_{B(0,1)} 1 \, dx + \int_{\mathbb{R}^n \setminus B(0,1)} |x|^{\gamma p} \, dx \right)^{1/p} \\ &= \left(|B(0,1)| + \int_1^\infty \int_{\partial B(0,r)} r^{\gamma p} \, dS(x) \, dr \right)^{1/p} \end{aligned}$$

(since $|\partial B(0,r)| = c_n r^{n-1}$ for some $c_n > 0$)

$$= \left(|B(0,1)| + \int_1^\infty c_n r^{\gamma p + n - 1} \, dr \right)^{1/p},$$

which is finite as long as $\gamma p + n < 0$, i.e. $\gamma < -n/p$. Likewise, we find that $\partial v / \partial x_i \in L^p(\mathbb{R}^n)$ if and only if $\gamma - 1 < -n/p$. Thus, $v \in W^{1,p}(\mathbb{R}^n)$ as long as $\gamma < -n/p$. We cannot hope for $v \in W^{2,p}$, as $\partial v / \partial x_i$ is too discontinuous (try to compute a second weak derivative of v !).

The function w is C^∞ , but does not lie in any other L^p space than L^∞ . Its partial derivatives are $\partial w / \partial x_i(x) = 2x_i \cos(|x|^2)$, which does not lie in L^∞ . Thus, w does not lie in any Sobolev space.

3. Let $\Omega \subset \mathbb{R}^n$ be open and $p \in [1, \infty]$. Let $u \in L^p(\mathbb{R}^n)$ be a discontinuous function, and let $u_\varepsilon := \omega_\varepsilon * u$ be its mollification. Explain why $\{u_\varepsilon\}_{\varepsilon>0}$ cannot converge to u in the L^∞ norm as $\varepsilon \rightarrow 0$, even when $u \in L^\infty(\mathbb{R}^n)$.

Hint. Show first that each function u_ε is uniformly continuous. You will need Hölder's inequality and the translational continuity of functions in $L^p(\mathbb{R}^n)$ (valid for $p \in [1, \infty)$):

$$\lim_{\substack{h \in \mathbb{R}^n \\ h \rightarrow 0}} \|u - u(\cdot + h)\|_{L^p(\mathbb{R}^n)} = 0.$$

Solution: Each function u_ε is uniformly continuous, and L^∞ convergence is equivalent to uniform convergence; the conclusion now follows from the fact that a uniformly convergent sequence of uniformly continuous functions will converge to a continuous function—contradicting the discontinuity of the limit u . (Note that uniform continuity is not a necessary ingredient here—uniform convergence of *continuous* functions implies the limit is *continuous*, which would also lead to a contradiction.)

To see that u_ε is uniformly continuous, assume first $p < \infty$, let $x, y \in \Omega$. Let $q \in [1, \infty]$ be the conjugate exponent of p , i.e. $1/p + 1/q = 1$. Then

$$\begin{aligned} |u_\varepsilon(x) - u_\varepsilon(y)| &= \left| \int_{B(0, \varepsilon)} \omega_\varepsilon(z)(u(x-z) - u(y-z)) dz \right| \\ &= \left| \int_{B(x, \varepsilon)} \omega_\varepsilon(x-z)(u(z) - u(y-x+z)) dz \right| \\ &\leq \|\omega_\varepsilon(x - \cdot)\|_{L^q(B(x, \varepsilon))} \|u - u(y-x + \cdot)\|_{L^p(B(x, \varepsilon))} \\ &\leq \|\omega_\varepsilon\|_{L^q(\mathbb{R}^n)} \|u - u(h + \cdot)\|_{L^p(\mathbb{R}^n)} \end{aligned}$$

where $h := y - x$. By letting $|h|$ be small enough and using the translational continuity of u in L^p , we find that the above can be made arbitrarily small. If $p = \infty$ then, since u automatically also lies in $L^1(B)$ for any bounded set B , we could apply the above with $p = 1$ and $\mathbb{R}^n$ replaced by $B(x, \varepsilon)$.

To see that a uniformly convergent sequence of uniformly continuous functions will converge to a continuous function, let $x \in \Omega$ and $\delta > 0$. Let $\varepsilon_0 > 0$ be such that $\|u_\varepsilon - u\|_{L^\infty} < \delta$ for all $\varepsilon \leq \varepsilon_0$, and let $r_0 > 0$ be such that $|y - x| < r_0$ implies $|u_{\varepsilon_0}(y) - u_{\varepsilon_0}(x)| < \delta$ (exists by uniform continuity). Then

$$\begin{aligned} |u(x) - u(y)| &\leq |u(x) - u_{\varepsilon_0}(x)| + |u_{\varepsilon_0}(x) - u_{\varepsilon_0}(y)| + |u_{\varepsilon_0}(y) - u(y)| \\ &\leq 3\delta \end{aligned}$$

whenever $|x - y| < r_0$. Thus, u is continuous.