

Problem sheet 3 – Solutions

MAT4305

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In this problem set we will use the Sobolev space theory that we have established so far to prove existence and uniqueness of a solution of a PDE. At the moment the type of equations that we can treat is quite limited, and the tools developed later will enable treatment of a much larger class of so-called elliptic equations. *You will find a collection of hints on page 4.*

Let $n \in \mathbb{N}$, $f \in L^2(\mathbb{R}^n) \cap C(\mathbb{R}^n)$, $c \in (0, \infty)$ and consider the following PDE:

$$-\Delta u + cu = f \quad \text{in } \mathbb{R}^n. \quad (1)$$

Here, Δ is the Laplace operator, $\Delta u = \sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2}$.

Definition. We say that $u \in C^2(\mathbb{R}^n)$ is a *classical solution* of (1) if it satisfies (1) at every $x \in \mathbb{R}^n$.

1. Show that every classical solution satisfies

$$\int_{\mathbb{R}^n} Du \cdot D\varphi + cu\varphi \, dx = \int_{\mathbb{R}^n} \varphi f \, dx \quad (2)$$

for all $\varphi \in C_c^\infty(\mathbb{R}^n)$.

2. Conversely, show that if $u \in C^2(\mathbb{R}^n)$ satisfies (2) for all $\varphi \in C_c^\infty(\mathbb{R}^n)$, then it is a classical solution of (1).

3. Although the above derivation assumes that u and φ are twice differentiable, (2) still makes sense for $u, \varphi \in H^1(\mathbb{R}^n) := W^{1,2}(\mathbb{R}^n)$, if the derivatives are interpreted as weak derivatives. Show that if $u \in H^1(\mathbb{R}^n)$ satisfies (2) for all $\varphi \in C_c^\infty(\mathbb{R}^n)$, then it also satisfies (2) for all $\varphi \in H^1(\mathbb{R}^n)$.

Definition. We say that $u \in H^1(\mathbb{R}^n)$ is a *weak solution* of (1) if it satisfies (2) for all $\varphi \in H^1(\mathbb{R}^n)$.

4. Let A and b be the functionals

$$A(u, \varphi) := \int_{\mathbb{R}^n} Du \cdot D\varphi + cu\varphi \, dx, \quad b(\varphi) := \int_{\mathbb{R}^n} \varphi f \, dx, \quad u, \varphi \in H^1(\mathbb{R}^n).$$

Thus, u is a weak solution if and only if $A(u, \varphi) = b(\varphi)$ for all $\varphi \in H^1(\mathbb{R}^n)$.

Show that b is a continuous linear functional on $H^1(\mathbb{R}^n)$, and that A is an inner product on $H^1(\mathbb{R}^n)$ which is equivalent to the “canonical” inner product

$$(u, v)_{H^1(\mathbb{R}^n)} := \int_{\mathbb{R}^n} uv \, dx + \sum_{i=1}^n \int_{\mathbb{R}^n} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} \, dx.$$

5. Apply Riesz' representation theorem for Hilbert spaces to show that there exists a unique $u \in H^1(\mathbb{R}^n)$ such that $A(u, \varphi) = b(\varphi)$ for all $\varphi \in H^1(\mathbb{R}^n)$.
6. Conclude that (1) has a unique weak solution for every $f \in L^2(\mathbb{R}^n)$.
7. Show that if $u \in H^1(\mathbb{R}^n)$ is a classical solution then it is a weak solution. Why can't we expect the opposite to be true?

Solution:

1. Multiply (1) by φ and integrate over $x \in \mathbb{R}^n$:

$$\int_{\mathbb{R}^n} \varphi f \, dx = \int_{\mathbb{R}^n} -\varphi \Delta u + c\varphi u \, dx = - \sum_{i=1}^n \int_{\mathbb{R}^n} \varphi \frac{\partial}{\partial x_i} \frac{\partial u}{\partial x_i} \, dx + c \int_{\mathbb{R}^n} u \varphi \, dx$$

(integrate by parts, using the fact that $\varphi = 0$ outside of a bounded set)

$$\begin{aligned} &= \sum_{i=1}^n \int_{\mathbb{R}^n} \frac{\partial \varphi}{\partial x_i} \frac{\partial u}{\partial x_i} \, dx + c \int_{\mathbb{R}^n} u \varphi \, dx \\ &= \int_{\mathbb{R}^n} D\varphi \cdot Du + cu\varphi \, dx. \end{aligned}$$

2. Start with (2) and integrate by parts backwards to get

$$\int_{\mathbb{R}^n} \varphi f \, dx = \int_{\mathbb{R}^n} -\varphi \Delta u + c\varphi u \, dx,$$

or equivalently,

$$\int_{\mathbb{R}^n} \varphi (f + \Delta u - cu) \, dx = 0.$$

(The function u is C^2 , so Δu makes sense.) Since this holds for all test functions φ , and the expression in the parentheses is continuous, we must have

$$f + \Delta u - cu \equiv 0,$$

which is precisely (1).

3. We first note that if (2) holds for all $\varphi \in C_c^\infty(\mathbb{R}^n)$, then it also holds for all $\varphi \in C^\infty(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. Indeed, let $\chi_M \in C_c^\infty(\mathbb{R}^n)$ for $M > 0$ satisfy $\chi_M(x) \equiv 1$ for all $|x| \leq M$ and $\chi_M(x) \equiv 0$ for all $|x| \geq M+1$ (see problem set 1 for the construction of such a function). If $\varphi \in C^\infty(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ then $\varphi_M := \varphi \chi_M \in C_c^\infty(\mathbb{R}^n)$ and $\varphi_M \rightarrow \varphi$ as $M \rightarrow \infty$ both in supremum norm and in $L^2(\mathbb{R}^n)$. Inserting φ_M in (2) and letting $M \rightarrow \infty$ yields the desired claim; for instance,

$$\int_{\mathbb{R}^n} \varphi_M f \, dx = \int_{\mathbb{R}^n} \varphi f \, dx + \int_{\mathbb{R}^n} (\varphi_M - \varphi) f \, dx,$$

and the last term can be bounded by $\|\varphi_M - \varphi\|_{L^2} \|f\|_{L^2} \rightarrow 0$ using Hölder's inequality.

Let $\varphi \in H^1(\mathbb{R}^n)$ and let $\varphi_\varepsilon := \omega_\varepsilon * \varphi$, the mollification of φ . Then $\varphi_\varepsilon \in C^\infty \cap L^2$ and $\varphi_\varepsilon \rightarrow \varphi$ in L^2 . Inserting φ_ε into (2), letting $\varepsilon \rightarrow 0$ and using a similar argument as above now yields the claim.

4. b is well-defined since $\int_{\mathbb{R}^n} |\varphi f| dx \leq \|\varphi\|_{L^2} \|f\|_{L^2} < \infty$, so φf is indeed integrable. It is clear that b is linear, by the linearity of the integral. Last, the above estimate yields

$$b(\varphi) \leq \int_{\mathbb{R}^n} |\varphi f| dx \leq C \|\varphi\|_{L^2} \leq C \|\varphi\|_{H^1}$$

where $C = \|f\|_{L^2}$. Hence, b is continuous.

The form A is clearly symmetric, is bilinear (i.e., linear in each argument separately), and if $u \neq 0$ then $A(u, u) = \|Du\|_{L^2}^2 + c\|u\|_{L^2}^2 > 0$. To see that A is equivalent to the canonical inner product, let $C_1 = \min(1, c)$ and $C_2 = \max(1, c)$ to get

$$\begin{aligned} C_1(u, u)_{H^1} &= \int_{\mathbb{R}^n} C_1 |Du|^2 + C_1 u^2 dx \leq \int_{\mathbb{R}^n} |Du|^2 + cu^2 dx = A(u, u) \\ C_2(u, u)_{H^1} &= \int_{\mathbb{R}^n} C_2 |Du|^2 + C_2 u^2 dx \geq \int_{\mathbb{R}^n} |Du|^2 + cu^2 dx = A(u, u). \end{aligned}$$

5. The functional b is continuous with respect to the $\|\cdot\|_{H^1}$ norm, so it is also continuous with respect to the norm $u \mapsto \sqrt{A(u, u)}$ (since, by the previous problem, these two norms are equivalent). Riesz' representation theorem then states that there is a unique element $u \in H^1$ such that b can be written as $\varphi \mapsto A(u, \varphi)$, which is what we wanted.
6. The equality $A(u, \varphi) = b(\varphi)$ for all $\varphi \in H^1$ is precisely our definition of being a weak solution.
7. By Problems 1 and 3, classical solutions are also weak solutions. We cannot expect the opposite: If $f \in L^2$ is discontinuous then there is a unique weak solution u , but this function cannot be a classical solution — by Problem 2 we would need $u \in C^2$, but then (1) states that a continuous function (the left-hand side) equals a discontinuous function (the right-hand side).

Hints

1. Multiply (1) by φ , integrate over $x \in \mathbb{R}^n$ and integrate by parts.
2. Do the same as in (a), but in reverse.
3. Show first that (2) being true for all $\varphi \in C_c^\infty$ implies that it holds for all $\varphi \in C^\infty \cap L^2$. To this end, multiply φ by a “cut off” function χ_M which is 1 on a ball of radius M and 0 outside the ball of radius $M + 1$; then let $M \rightarrow \infty$.

Let $\varphi_\varepsilon = \omega_\varepsilon * \varphi$, the mollification of φ . Insert φ_ε in (2), and then let $\varepsilon \rightarrow 0$.

4. For b you need to show that (i) b is finite for every φ , (ii) that b is linear, and that (iii) b is continuous, which is the same as the existence of a $C > 0$ such that $b(\varphi) \leq C \|\varphi\|_{H^1}$ for every φ .

Check e.g. Wikipedia for the definition of an inner product. By “equivalent” we mean that there are $C_1, C_2 > 0$ such that

$$C_1(u, u)_{H^1(\mathbb{R}^n)} \leq A(u, u) \leq C_2(u, u)_{H^1(\mathbb{R}^n)} \quad \forall u \in H^1(\mathbb{R}^n).$$