

Problem sheet 4

MAT4305

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You will find a collection of hints on page 2.

1. (Problem 5.10.17 in Evans) Prove the chain rule for Sobolev functions: If $F: \mathbb{R} \rightarrow \mathbb{R}$ is C^1 and has bounded derivative, $\Omega \subset \mathbb{R}^n$ is open and bounded, $p \in [1, \infty]$ and $u \in W^{1,p}(\Omega)$, then the composition $v := F(u)$ lies in $W^{1,p}(\Omega)$, and

$$v_{x_i} = F'(u)u_{x_i} \quad \text{for } i = 1, \dots, n.$$

2. (Problem 5.10.9 in Evans) Let $\Omega \subset \mathbb{R}^n$ be open.

(a) Prove the interpolation inequality

$$\|Du\|_{L^2(\Omega)} \leq C \|u\|_{L^2(\Omega)}^{1/2} \|D^2u\|_{L^2(\Omega)}^{1/2} \quad (1)$$

for all $u \in C_c^\infty(\Omega)$. (Here, $\|D^2u\|_{L^2} = \left(\int_\Omega \sum_{i,j=1}^n |u_{x_i x_j}(x)|^2 dx\right)^{1/2}$.)

(b) Under the additional conditions that Ω is bounded and $\partial\Omega$ is C^1 , prove that (1) also holds for all $u \in H^2(\Omega) \cap H_0^1(\Omega)$.

3. (Problem 5.10.11 in Evans) Let $\Omega \subset \mathbb{R}^n$ be open and connected, let $p \in [1, \infty]$, and let $u \in W^{1,p}(\Omega)$ satisfy

$$Du(x) = 0 \quad \text{for a.e. } x \in \Omega.$$

Show that $u(x) = c$ for a.e. $x \in \Omega$ for some constant $c \in \mathbb{R}$.

4. (Problem 5.10.13 in Evans) Give an example of an open, bounded domain $\Omega \subset \mathbb{R}^n$ and a $u \in W^{1,\infty}(\Omega)$ which is not Lipschitz continuous.

Remark. Morrey's inequality guarantees that if $\Omega \subset \mathbb{R}^n$ is open, bounded and has C^1 boundary, then any $u \in W^{1,\infty}(\Omega)$ is Lipschitz continuous (i.e., lies in $C^{0,1}(\Omega)$). This example shows that we cannot drop the smoothness condition on $\partial\Omega$.

Hints

1. Separate the proof into the following steps:
 - (a) Show that v and the functions $w_i := F'(u)u_{x_i}$ all lie in $L^p(\Omega)$.
 - (b) Pick $u^1, u^2, \dots \in C^\infty(\Omega)$ converging to u in $W^{1,1}(V)$ on every $V \Subset \Omega$.
 - (c) Explain why $F'(u^m) \rightarrow F'(u)$ pointwise a.e., and $F(u^m) \rightarrow F(u)$ in $L^1(\Omega)$.
 - (d) Show that $w_i = v_{x_i}$ by letting $\varphi \in C_c^\infty(\Omega)$, $V = \text{supp } \varphi$, and computing directly.
2.
 - (a) Begin with $\|Du\|_{L^2}$ and integrate by parts.
 - (b) Since $u \in H_0^1(\Omega)$, there are $v_1, v_2, \dots \in C_c^\infty(\Omega)$ such that $v_m \rightarrow u$ in $H_0^1(\Omega)$. Since $u \in H^2(\Omega)$, we can extend u to all of $\mathbb{R}^n$, then mollify it, then restrict the mollifications to Ω to end up with $w_1, w_2, \dots \in C^\infty(\overline{\Omega})$ such that $w_m \rightarrow u$ in $H^2(\Omega)$. Finally, use the idea $|Du(x)|^2 \approx Dv_m(x) \cdot Dw_m(x)$. (Note: I wasn't able to come up with an argument which uses (a).)
3. Extend u by zero outside of Ω and mollify it. Use the second approximation theorem to say something about the convergence of the mollifications in each set $\Omega_\delta := \{x \in \Omega : \text{dist}(x, \partial\Omega) > \delta\}$. What can you say about u_ϵ in the sets Ω_δ ?
4. Let Ω be the unit ball with a slit removed.