

Problem sheet 4 – Solutions

MAT4305

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1. (*Problem 5.10.17 in Evans*) Prove the chain rule for Sobolev functions: If $F: \mathbb{R} \rightarrow \mathbb{R}$ is C^1 and has bounded derivative, $\Omega \subset \mathbb{R}^n$ is open and bounded, $p \in [1, \infty]$ and $u \in W^{1,p}(\Omega)$, then the composition $v := F(u)$ lies in $W^{1,p}(\Omega)$, and

$$v_{x_i} = F'(u)u_{x_i} \quad \text{for } i = 1, \dots, n.$$

Solution: Let $M := \|F'\|_{L^\infty(\mathbb{R})}$, which is bounded, by assumption. Let $w_i(x) := F'(u)u_{x_i}$. We first claim that v and $w_1, \dots, w_n$ lie in $L^p(\Omega)$. Indeed, F is C^1 , so by the mean value theorem,

$$|v(x)| = |F(u(x))| \leq |F(u(x)) - F(0)| + |F(0)| = |F'(\xi)||u(x)| + |F(0)| \leq M|u(x)| + |F(0)|,$$

so by the triangle inequality,

$$\|v\|_{L^p(\Omega)} \leq M\|u\|_{L^p(\Omega)} + |F(0)||\Omega| < \infty.$$

Furthermore,

$$\|w_i\|_{L^p(\Omega)} \leq M\|u_{x_i}\|_{L^p(\Omega)} < \infty.$$

This proves the claim.

Assume first $p < \infty$. By the first approximation theorem, there are $u^1, u^2, \dots \in C^\infty(\Omega)$ such that $u^m \rightarrow u$ as $m \rightarrow \infty$ in $W^{1,1}(V)$ for any $V \Subset \Omega$. In particular, $u^m \rightarrow u$ in $L^1(V)$; since $|F(u^m) - F(u)| \leq M|u^m - u|$, we have that also $F(u^m) \rightarrow F(u) = v$ in $L^1(\Omega)$. Moreover, since $u^m \rightarrow u$ pointwise a.e. in V and since F' is continuous, we also have $F'(u^m) \rightarrow F'(u)$ pointwise a.e. in V .

We claim that w_i is the i -th weak derivative of v . Let $\varphi \in C_c^\infty(\Omega)$. Setting $V = \text{supp } \varphi$, we find that

$$\begin{aligned} \int_{\Omega} v \varphi_{x_i} dx - \left(- \int_{\Omega} w_i \varphi dx \right) &= \lim_{m \rightarrow \infty} \int_V F(u^m) \varphi_{x_i} dx + \int_V F'(u) u_{x_i} \varphi dx \\ &= - \lim_{m \rightarrow \infty} \int_V F'(u^m) u_{x_i}^m \varphi dx + \int_V F'(u) u_{x_i} \varphi dx \\ &= \lim_{m \rightarrow \infty} \left(\int_V (F'(u) - F'(u^m)) u_{x_i} \varphi dx + \int_V F'(u^m) (u_{x_i} - u_{x_i}^m) \varphi dx \right) \\ &= 0, \end{aligned}$$

where we have used Lebesgue's dominated convergence theorem for the first term, and the facts that $F'(u^m)$ is bounded and $u_{x_i}^m \rightarrow u_{x_i}$ in L^1 for the second term. This concludes the proof.

If $p = \infty$ then $u \in W^{1,q}(\Omega)$ for all $q \in [1, \infty)$, since Ω is bounded, so we can apply the same argument for e.g. $p = 1$.

2. (Problem 5.10.9 in Evans) Let $\Omega \subset \mathbb{R}^n$ be open.

(a) Prove the interpolation inequality

$$\|Du\|_{L^2(\Omega)} \leq C \|u\|_{L^2(\Omega)}^{1/2} \|D^2u\|_{L^2(\Omega)}^{1/2} \quad (1)$$

for all $u \in C_c^\infty(\Omega)$. (Here, $\|D^2u\|_{L^2} = (\int_\Omega \sum_{i,j=1}^n |u_{x_i x_j}(x)|^2 dx)^{1/2}$.)

(b) Under the additional conditions that Ω is bounded and $\partial\Omega$ is C^1 , prove that (1) also holds for all $u \in H^2(\Omega) \cap H_0^1(\Omega)$.

Solution:

(a) We have

$$\begin{aligned} \|Du\|_{L^2(\Omega)}^2 &= \sum_{i=1}^n \int_\Omega u_{x_i}(x) u_{x_i}(x) dx = - \sum_{i=1}^n \int_\Omega u(x) u_{x_i x_i}(x) dx \\ &\leq \|u\|_{L^2} \left(\int_\Omega \left| \sum_{i=1}^n u_{x_i x_i}(x) \right|^2 dx \right)^{1/2} \leq n \|u\|_{L^2} \|D^2u\|_{L^2}, \end{aligned}$$

where we have first integrated by parts, then applied Hölder's inequality, and then used the inequality $|\sum_{i=1}^m a_i|^2 \leq m \sum_{i=1}^m |a_i|^2$. Taking square roots on both sides yields (1) with $C = \sqrt{n}$.

(b) Let $u \in H^2(\Omega) \cap H_0^1(\Omega)$. Since $u \in H_0^1(\Omega)$, there are $v_1, v_2, \dots \in C_c^\infty(\Omega)$ such that $v_m \rightarrow u$ in $H^1(\Omega)$ as $m \rightarrow \infty$. On the other hand, since $u \in H^2(\Omega)$, we can use the extension theorem to see that there is some $\tilde{u} \in H^2(\mathbb{R}^n)$ with compact support such that $\tilde{u}|_\Omega = u$ and $\|\tilde{u}\|_{H^2(\mathbb{R}^n)} \leq C \|u\|_{H^2(\Omega)}$, where $C > 0$ is independent of u . Letting $\tilde{u}_m := \tilde{u} * \omega_{1/m} \in C_c^\infty(\mathbb{R}^n)$, we have $\tilde{u}_m \rightarrow \tilde{u}$ in $H^2(\mathbb{R}^n)$, so in particular, if $w_m := \tilde{u}_m|_\Omega \in C^\infty(\overline{\Omega})$, then $w_m \rightarrow u$ in $H^2(\Omega)$. We get

$$\begin{aligned} \|Du\|_{L^2(\Omega)}^2 &= \lim_{m \rightarrow \infty} \int_\Omega Dv_m \cdot Dw_m dx \\ &= \lim_{m \rightarrow \infty} \left(\int_{\partial\Omega} \underbrace{v_m}_{=0} Dw_m \cdot \nu dS(x) - \int_\Omega v_m \Delta w_m dx \right) \\ &\leq \lim_{m \rightarrow \infty} C \|v_m\|_{L^2(\Omega)} \|D^2w_m\|_{L^2(\Omega)} = C \|u\|_{L^2} \|D^2u\|_{L^2}, \end{aligned}$$

as in the previous proof.

3. (Problem 5.10.11 in Evans) Let $\Omega \subset \mathbb{R}^n$ be open and connected, let $p \in [1, \infty]$, and let $u \in W^{1,p}(\Omega)$ satisfy

$$Du(x) = 0 \quad \text{for a.e. } x \in \Omega.$$

Show that $u(x) = c$ for a.e. $x \in \Omega$ for some constant $c \in \mathbb{R}$.

Solution: Extend u by zero outside of Ω and let $u_\varepsilon := u * \omega_\varepsilon$. Let $\Omega_\delta := \{x \in \Omega : \text{dist}(x, \partial\Omega) > \delta\}$. We then know from the second approximation theorem that $u_\varepsilon \rightarrow u$ as $\varepsilon \rightarrow 0$ in $W^{1,p}(\Omega_\delta)$ for every $\delta > 0$. Moreover, if $\varepsilon \leq \delta$ then $Du_\varepsilon = Du * \omega_\varepsilon$, which is constant in Ω_δ . But u_ε is C^∞ , so $u_\varepsilon \equiv \text{const.}$ in each connected component of Ω_δ . Letting $\varepsilon \rightarrow 0$, we find that $u \equiv \text{const.}$ in each connected component of Ω_δ . Since any disconnected components in Ω_δ are eventually connected by letting δ be small enough, we find that $u \equiv \text{const.}$ in each connected component of Ω — that is, $u \equiv \text{const.}$ in Ω (since Ω consists of only one component).

4. (Problem 5.10.13 in Evans) Give an example of an open, bounded domain $\Omega \subset \mathbb{R}^n$ and a $u \in W^{1,\infty}(\Omega)$ which is not Lipschitz continuous.

Remark. Morrey's inequality guarantees that if $\Omega \subset \mathbb{R}^n$ is open, bounded and has C^1 boundary, then any $u \in W^{1,\infty}(\Omega)$ is Lipschitz continuous (i.e., lies in $C^{0,1}(\Omega)$). This example shows that we cannot drop the smoothness condition on $\partial\Omega$.

Solution: Let $\Omega = B^0(0, 1) \setminus \{(x_1, 0) : x_1 \geq 0\}$ and define

$$v(\theta, r) := r^2\theta, \quad u(x) := v(\theta(x), r(x))$$

where $(\theta(x), r(x))$ are the polar coordinates of $x \in \mathbb{R}^2$ (with $\theta = 0$ being the positive x_1 -axis). It is clear that u is not Lipschitz continuous, as it has a discontinuity along $(x_1, 0)$ for $x_1 > 0$. Indeed, if $x_1 \in (0, 1)$ then $(x_1, \pm h) \in \Omega$ for any sufficiently small $h > 0$, but

$$\lim_{h \rightarrow 0} \frac{|u(x_1, h) - u(x_1, -h)|}{|(x_1, h) - (x_1, -h)|} = \lim_{h \rightarrow 0} \frac{|0 - (2\pi x_1^2)|}{2h} = \infty.$$

We claim that $u \in W^{1,\infty}(\Omega)$ — in fact, $u \in C^1(\Omega)$ with bounded derivative. To this end, we recall that $r(x) = \sqrt{x_1^2 + x_2^2}$, so

$$\frac{\partial r(x)}{\partial x_i} = \frac{x_i}{r(x)}, \quad i = 1, 2.$$

Moreover,

$$r(x) \cos \theta(x) = x_1, \quad r(x) \sin \theta(x) = x_2,$$

so taking partial derivatives,

$$\begin{aligned} 0 &= \frac{\partial r(x)}{\partial x_2} \cos \theta(x) - r(x) \sin \theta(x) \frac{\partial \theta(x)}{\partial x_2} = \frac{x_2}{r(x)} \cos \theta(x) - x_2 \frac{\partial \theta(x)}{\partial x_2} \\ &\Rightarrow \frac{\partial \theta}{\partial x_2}(x) = \frac{\cos(\theta(x))}{r(x)}. \end{aligned}$$

Likewise,

$$\frac{\partial \theta}{\partial x_1}(x) = -\frac{\sin \theta(x)}{r(x)}.$$

It follows that

$$\begin{aligned}\frac{\partial u}{\partial x_1}(x) &= 2r(x) \frac{\partial r(x)}{\partial x_1} \theta(x) + r(x)^2 \frac{\partial \theta(x)}{\partial x_1} \\ &= 2r(x) \frac{x_1}{r(x)} \theta(x) - r(x)^2 \frac{\sin \theta(x)}{r(x)} \\ &= 2x_1 \theta(x) - x_2\end{aligned}$$

which is a bounded function on Ω . Likewise, we get

$$\frac{\partial u}{\partial x_2}(x) = 2x_2 \theta(x) + x_1$$

which is also bounded. We conclude that both u and Du lie in L^∞ , whence $u \in W^{1,\infty}(\Omega)$.