

Problem sheet 5 – Solutions

MAT4305

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1. For each of the following PDEs, determine if the problem is

- (i) linear or nonlinear
- (ii) first-, second-, or higher-order
- (iii) homogeneous or inhomogeneous
- (iv) elliptic or non-elliptic.

$$-2u_{x_1x_2} + 5|x|^2u_{x_1} = 3 \quad (1)$$

$$\sum_{i=1}^n u_{x_i x_i} - u + \cos(x_1 x_2) = 0 \quad (2)$$

$$-2u_{x_1x_1} + u_{x_1x_2x_3} + 4u_{x_3} + u^2 = 0 \quad (3)$$

$$-u_{x_1x_1} + 2u_{x_1x_2} - 5u_{x_2x_2} + 8x_1u_{x_2} = 3(x_1 - x_2)^2 \quad (4)$$

$$-\sum_{i,j=1}^3 u_{x_i x_j} = 5 \sum_{i=1}^3 x_i u_{x_i} \quad (5)$$

$$u_t = u_{x_1x_1} + u_{x_2x_2}. \quad (6)$$

Remark. See the hints for the definitions of these terms. Note that in all such classifications of PDEs, it is implicit in the definition that the PDE *can* be written in a certain way, even if it isn't already written in that way. For instance, the PDE $u_{x_1x_1} + u_{x_2x_2} = 0$ is homogeneous, even though it can be written in the “seemingly non-homogeneous” form $u_{x_1x_1} + u_{x_2x_2} + |x|^2 = |x|^2$.

Solution:

- (1) This PDE is linear, second-order and inhomogeneous. It can be written in the abstract formulation

$$Lu = -\sum_{i,j} (a^{i,j} u_{x_i})_{x_j} + \sum_i b^i u_{x_i} + cu$$

for $b^i(x) = 5|x|^2$, $c \equiv 0$, $a^{1,2} = 2$ and $a^{i,j} = 0$ otherwise. The matrix $A = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$ is not positive definite, since e.g. $\xi^T A \xi = -1$ when $\xi = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Therefore, the problem is not elliptic.

- (2) This PDE is linear, second-order and inhomogeneous. We can write it in the above form with $a^{i,j} = \delta_{i,j}$ (the Kronecker delta), $b^i \equiv 0$, $c \equiv 1$ and $f(x) = \cos(x_1 x_2)$. The matrix $(a^{i,j})_{i,j}$ is the identity matrix, which is positive definite, so the PDE is elliptic.
- (3) The PDE is nonlinear (because of the term u^2), third-order and homogeneous. We do not define ellipticity for third-order PDEs.
- (4) The PDE is linear, second-order and non-homogeneous. It can be written in the above manner with $b^i = c \equiv 0$ and $(a^{i,j})_{i,j} = A := \begin{pmatrix} 1 & -2 \\ 0 & 5 \end{pmatrix}$. We have

$$\xi^T A \xi = \xi_1^2 - 2\xi_1 \xi_2 + 5\xi_2^2 = (\xi_1 - \xi_2)^2 + 4\xi_2^2,$$

which is positive for all $\xi \neq 0$.

- (5) This PDE is linear, second-order and homogeneous. It can be written in the above manner with $b^i(x) = x_i$, $c \equiv 0$ and $(a^{i,j})_{i,j} = A := \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$. If e.g. $\xi =$

$$\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \text{ then } \xi^T A \xi = 0, \text{ so the PDE is not elliptic.}$$

- (6) We interpret the equation as searching for some $u = u(t, x)$ satisfying the PDE; we may think of t as the zeroth component of u , so $u = u(x_0, x_1, x_2)$. The PDE is linear, second-order and homogeneous. The second-order part can be written as $-\sum_{i,j=0}^2 a^{i,j} u_{x_i x_j}$, where $(a^{i,j})_{i,j} = A := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. The equation is *not* elliptic, since e.g. $\xi^T A \xi = 0$ when $\xi = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. (This is the heat equation, which is *parabolic*, not elliptic.)

2. Let $Lu = -\sum_{i,j} (a^{i,j} u_{x_i})_{x_j} + \sum_i b^i u_{x_i} + cu$ be a second-order differential operator such that

$$\sum_{i,j} a^{i,j}(x) \xi_i \xi_j \leq -\theta |\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^n \text{ and } x \in \Omega \quad (7)$$

for some $\theta > 0$.

- (a) Show that $-L$ is elliptic.

- (b) Show that the problem

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad (8)$$

has a unique weak solution for any $f \in L^2(\Omega)$ if and only if

$$\begin{cases} -Lv = g & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases} \quad (9)$$

has a unique weak solution for any $g \in L^2(\Omega)$.

Remark. When writing a PDE in the abstract form $Lu = f$, there is ambiguity in defining L and f , since we can always multiply the equation by -1 . This problem shows that doing so does not influence the well-posedness (the existence, uniqueness and stability of solutions) of the problem.

Solution:

- (a) The linear operator $-L$ has second-order coefficients $-a^{i,j}$, whence (7) implies that $-L$ is elliptic.
- (b) If u solves (8) then $v = -u$ solves (9) with $g = -f$. Hence, existence of one equation implies existence for the other, and likewise for uniqueness.

3. A square matrix $A \in \mathbb{R}^{n \times n}$ is symmetric if $A^\top = A$. The *symmetric part* (or the *symmetrization*) of a matrix $A \in \mathbb{R}^{n \times n}$ is the matrix $A_S := \frac{1}{2}(A + A^\top)$.

- (a) Show that A_S is always symmetric.
- (b) Find the *anti-symmetric part* of A , that is, the matrix $A_A \in \mathbb{R}^{n \times n}$ so that $A = A_S + A_A$. Show that A_A is anti-symmetric (i.e., $A_A^\top = -A_A$).
- (c) A matrix A is *positive definite* if

$$\xi^\top A \xi > 0 \quad \text{for all nonzero } \xi \in \mathbb{R}^n.$$

(It can be shown that this is equivalent to the existence of some $\theta > 0$ such that $\xi^\top A \xi \geq \theta |\xi|^2$ for all $\xi \in \mathbb{R}^n$.) Show that A is positive definite if and only if A_S is.

- (d) Let $Lu = -\sum_{i,j} (a^{ij} u_{x_i})_{x_j}$ be the second-order part of a linear differential operator. Let $A_S(x)$ be the symmetric part of the matrix $A(x) = (a^{i,j}(x))_{i,j=1,\dots,n}$. Show that L is elliptic if and only if there is some $\theta > 0$ such that $\xi^\top A_S(x) \xi > \theta |\xi|^2$ for all $x \in \Omega$ and $\xi \in \mathbb{R}^n$.

Remark. Since $u_{x_i x_j} = u_{x_j x_i}$ for any i, j , there is ambiguity in how to write the second-order differential operator in the form $\sum_{i,j} (a^{ij} u_{x_i})_{x_j}$. This problem shows that it does not matter how we choose the coefficients $a^{i,j}$. In particular, we can always symmetrize the matrix $(a^{i,j})_{i,j}$ and work with that instead.

Solution:

- (a) We have

$$A_S^\top = \frac{1}{2}(A + A^\top)^\top = \frac{1}{2}(A^\top + A) = A_S.$$

- (b) If $A = A_S + A_A$ then $A_A = A - A_S = A - \frac{1}{2}(A + A^\top) = \frac{1}{2}(A - A^\top)$. We get

$$A_A^\top = \frac{1}{2}(A^\top - A) = -A_A,$$

so A_A is anti-symmetric.

- (c) We have

$$\xi^\top A_S \xi = \frac{1}{2} \xi^\top (A + A^\top) \xi = \frac{1}{2} (\xi^\top A \xi + \xi^\top A^\top \xi).$$

We note that $\xi^\top A^\top \xi$ is a one-by-one matrix (i.e., a number), so it is equal to its own

transpose $(\xi^\top A^\top \xi)^\top = \xi^\top A \xi$. Thus:

$$\xi^\top A_S \xi = \frac{1}{2}(\xi^\top A \xi + \xi^\top A \xi) = \xi^\top A \xi.$$

In particular, $\xi^\top A_S \xi > 0$ if and only if $\xi^\top A \xi > 0$, which proves the result.

- (d) We have $\xi^\top A_S(x) \xi = \xi^\top A(x) \xi$. The operator L is elliptic if and only if there is some $\theta > 0$ such that $\xi^\top A(x) \xi \geq \theta |\xi|^2$, which is the same as saying that $\xi^\top A_S(x) \xi \geq \theta |\xi|^2$.