

Problem sheet 6

MAT4305

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You will find a collection of hints on page 3.

1. Consider the PDE

$$\begin{cases} \sum_{i=1}^n u_{x_i x_i} - u + \cos(x_1 x_2) = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad (1)$$

where $\Omega = B^0(0, 1)$ is the open unit ball in $\mathbb{R}^n$. (This is equation (2) in Problem sheet 5.)

(a) Write (1) as

$$B[u, v] = (f, v)_{L^2} \quad \forall v \in H_0^1(\Omega) \quad (2)$$

for some $f \in L^2(\Omega)$ and a continuous, coercive bilinear form B .

(b) Show that (1) has a unique weak solution $u \in H_0^1(\Omega)$.

(c) Use the interior regularity theory (Section 6.3.1 in Evans) to show that $u \in H^m(V)$ for any $V \Subset \Omega$, for any $m \in \mathbb{N}$.

(d) Give a more direct, but informal, proof of the previous problem, as follows.

(i) Use continuity and coercivity of B to show that $\|u\|_{H^1(\Omega)} \leq \|f\|_{L^2(\Omega)}$.

(ii) Show that for any $k = 1, \dots, n$, the function u_{x_k} solves the same PDE (ignoring for the moment that u_{x_k} is not in H^1), with f_{x_k} in place of f . Deduce that $\|u_{x_k}\|_{H^1(\Omega)} \leq \|f_{x_k}\|_{L^2(\Omega)}$.

(iii) Iterate the above to find that $u \in H^m(\Omega)$ for any m , and $\|u\|_{H^{m+1}(\Omega)} \leq C \|f\|_{H^m(\Omega)}$ for some $C > 0$.

2. Consider the PDE

$$\begin{cases} -(1 + H(x))u_{x_1 x_1} + (x_2 - 2)u_{x_2 x_2} + u_{x_1 x_2} = |x| & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad (3)$$

where $\Omega = B^0(0, 1)$ is the open unit ball in $\mathbb{R}^2$ and

$$H(x) := \begin{cases} |x_1|^2 & \text{if } x_1 > 0 \\ 0 & \text{else.} \end{cases}$$

(a) Show that (3) is elliptic, and that it can be written in the form (2) for a continuous, coercive bilinear form B . Conclude that (3) has a unique weak solution $u \in H_0^1(\Omega)$.

(b) According to the interior regularity theory, what is the largest $m \in \mathbb{N}$ for which $u \in H_{\text{loc}}^m(\Omega)$?

3. Consider the same PDE as in (1), but with nontrivial boundary data

$$u = g \quad \text{on } \partial\Omega$$

for some $g: \partial\Omega \rightarrow \mathbb{R}$.

- (a) Outline how you would determine if this boundary value problem has a (unique) solution.
- (b) For each of the following choices of n and g , determine the largest $m \in \mathbb{N}$ for which the solution lies in $H_{\text{loc}}^m(\Omega)$.
 - (i) $n \in \mathbb{N}$, $g(x) = e^{x_2 - x_1}$.
 - (ii) $n = 3$, $g(x) = |x - x_0|^{7/3}$, where $x_0 = (1, 0, 0)$.

4. Let $\Omega \subset \mathbb{R}^3$ be open and bounded and let $f \in H^2(\Omega)$. Show that any weak solution $u \in H^1(\Omega)$ of

$$-\Delta u = f \quad \text{in } \Omega \tag{4}$$

is also a classical solution.

Hints

2. (a) You will need Poincaré's inequality for coercivity.
3. Try to find a function $\bar{g}: \overline{\Omega} \rightarrow \mathbb{R}$ so that $\bar{g}|_{\partial\Omega} = g$. Write $u = \bar{g} + w$. Find a boundary value problem that w should solve.
4. Use Morrey's inequality.