

Problem sheet 6 – Solutions

MAT4305

Ulrik Skre Fjordholm

Updated June 7, 2023

1. Consider the PDE

$$\begin{cases} \sum_{i=1}^n u_{x_i x_i} - u + \cos(x_1 x_2) = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad (1)$$

where $\Omega = B^0(0, 1)$ is the open unit ball in $\mathbb{R}^n$. (This is equation (2) in Problem sheet 5.)

(a) Write (1) as

$$B[u, v] = (f, v)_{L^2} \quad \forall v \in H_0^1(\Omega) \quad (2)$$

for some $f \in L^2(\Omega)$ and a continuous, coercive bilinear form B .

(b) Show that (1) has a unique weak solution $u \in H_0^1(\Omega)$.

(c) Use the interior regularity theory (Section 6.3.1 in Evans) to show that $u \in H^m(V)$ for any $V \Subset \Omega$, for any $m \in \mathbb{N}$.

(d) Give a more direct, but informal, proof of the previous problem, as follows.

- (i) Use continuity and coercivity of B to show that $\|u\|_{H^1(\Omega)} \leq \|f\|_{L^2(\Omega)}$.
- (ii) Show that for any $k = 1, \dots, n$, the function u_{x_k} solves the same PDE (ignoring for the moment that u_{x_k} is not in H^1), with f_{x_k} in place of f . Deduce that $\|u_{x_k}\|_{H^1(\Omega)} \leq \|f_{x_k}\|_{L^2(\Omega)}$.
- (iii) Iterate the above to find that $u \in H^m(\Omega)$ for any m , and $\|u\|_{H^{m+1}(\Omega)} \leq C \|f\|_{H^m(\Omega)}$ for some $C > 0$.

Solution:

(a) We get $B[u, v] = \sum_{i=1}^n \int_{\Omega} u_{x_i} v_{x_i} dx + \int_{\Omega} uv dx$ and $f(x) = \cos(x_1 x_2)$. The function f is continuous and bounded, and therefore lies in $L^2(\Omega)$. If $u \in H^1(\Omega)$ then

$$B[u, u] = \int_{\Omega} |Du|^2 + |u|^2 dx = \|u\|_{H^1}^2$$

so B is coercive, whereas

$$B[u, v] \leq \|Du\|_{L^2} \|Dv\|_{L^2} + \|u\|_{L^2} \|v\|_{L^2} \leq \|u\|_{H^1} \|v\|_{H^1},$$

so B is continuous.

(b) This follows from Lax–Milgram.

(c) The PDE has constant coefficients and the right-hand side f is C^∞ , so according to the interior regularity theorem, $u \in H_{\text{loc}}^m(\Omega)$ for any $m \in \mathbb{N}$.

(d) (i) We have

$$\|u\|_{H^1}^2 = B[u, u] = (f, u)_{L^2} \leq \|f\|_{L^2} \|u\|_{L^2} \leq \|f\|_{L^2} \|u\|_{H^1},$$

so dividing by $\|u\|_{H^1}$ on both sides yields the claim.

(ii) Informally differentiating (1) by x_k , we see that u_{x_k} solves the same PDE, with f_{x_k} in place of f . The estimate now follows since $f_{x_k} \in L^2$.

(iii) Iterating the above idea, we get $\|D^\alpha u\|_{H^1} \leq \|D^\alpha f\|_{L^2}$ for any multiindex α . Hence,

$$\|u\|_{H^{m+1}} = \sum_{|\alpha| \leq k} \|D^\alpha u\|_{L^2} \leq \sum_{|\alpha| \leq k} \|D^\alpha u\|_{H^1} \leq \sum_{|\alpha| \leq k} \|D^\alpha f\|_{L^2} = \|f\|_{H^m}.$$

2. Consider the PDE

$$\begin{cases} -((1 + H(x))u_{x_1})_{x_1} + ((x_2 - 2)u_{x_2})_{x_2} + u_{x_1 x_2} = |x| & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad (3)$$

where $\Omega = B^0(0, 1)$ is the open unit ball in $\mathbb{R}^2$ and

$$H(x) := \begin{cases} |x_1|^2 & \text{if } x_1 > 0 \\ 0 & \text{else.} \end{cases}$$

(a) Show that (3) is elliptic, and that it can be written in the form (2) for a continuous, coercive bilinear form B . Conclude that (3) has a unique weak solution $u \in H_0^1(\Omega)$.

(b) According to the interior regularity theory, what is the largest $m \in \mathbb{N}$ for which $u \in H_{\text{loc}}^m(\Omega)$?

Solution:

(a) The PDE can be written as

$$-\sum_{i,j=1}^2 (a^{i,j}(x)u_{x_i})_{x_j} = f$$

with

$$(a^{i,j}(x))_{i,j=1}^2 = A(x) := \begin{pmatrix} 1 + H(x) & -1/2 \\ -1/2 & 2 - x_2 \end{pmatrix}, \quad f(x) = |x|.$$

Then

$$\begin{aligned} \xi^\top A(x) \xi &= (1 + H(x))\xi_1^2 - \xi_1 \xi_2 + (2 - x_2)\xi_2^2 \\ &= \frac{1}{2}(\xi_1^2 + \xi_2^2) + \underbrace{\frac{1}{2}(\xi_1 - \xi_2)^2 + H(x)\xi_1^2 + (1 - x_2)\xi_2^2}_{\geq 0} \\ &\geq \frac{1}{2}|\xi|^2, \end{aligned}$$

so the equation is elliptic, with $\theta = \frac{1}{2}$. The weak formulation is (2) with

$$B[u, v] = \sum_{i,j} \int_{\Omega} a^{i,j}(x) u_{x_i} v_{x_j} dx,$$

which by similar estimates as above is continuous. For coercivity, we have

$$B[u, u] = \int_{\Omega} \sum_{i,j} a^{i,j} u_{x_i} u_{x_j} dx \geq \theta \int_{\Omega} |Du|^2 dx = \theta \|Du\|_{L^2}^2.$$

Using Poincaré's inequality for $u \in H_0^1(\Omega)$, we get

$$\|u\|_{H^1}^2 = \|u\|_{L^2}^2 + \|Du\|_{L^2}^2 \leq (1 + C) \|Du\|_{L^2}^2 = (1 + C) B[u, u],$$

so B is coercive. We conclude by Lax–Milgram that (3) has a unique weak solution in H_0^1 .

- (b) The coefficients $a^{i,j}$ lie in $C^1(\Omega)$ but not in $C^2(\Omega)$ (because the same is true for H). The function f has weak derivatives

$$f_{x_i}(x) = \frac{x_i}{|x|}, \quad f_{x_i x_j}(x) = \frac{\delta_{i,j} |x|^2 - x_i x_j}{|x|^3}.$$

The functions f and f_{x_i} are bounded, and therefore lie in $L^2(\Omega)$. On the other hand, $f_{x_i x_j}$ behaves like $|x|^{-1}$, which lies in $L^p(\Omega)$ only for $p < 2$. Hence, f lies in $H^1(\Omega)$ but not in $H^2(\Omega)$.

By the regularity theory, if $a^{ij} \in C^{k+1}$ and $f \in H^k$ then $u \in H_{\text{loc}}^{k+2}$. Thus, we can set $k = 0$, but not higher, so $u \in H_{\text{loc}}^2(\Omega)$.

3. Consider the same PDE as in (1), but with nontrivial boundary data

$$u = g \quad \text{on } \partial\Omega$$

for some $g: \partial\Omega \rightarrow \mathbb{R}$.

- Outline how you would determine if this boundary value problem has a (unique) solution.
- For each of the following choices of n and g , determine the largest $m \in \mathbb{N}$ for which the solution lies in $H_{\text{loc}}^m(\Omega)$.
 - $n \in \mathbb{N}$, $g(x) = e^{x_2 - x_1}$.
 - $n = 3$, $g(x) = |x - x_0|^{7/3}$, where $x_0 = (1, 0, 0)$.

Solution:

- (a) We first seek a function $\bar{g}: \bar{\Omega} \rightarrow \mathbb{R}$ so that $\bar{g}|_{\partial\Omega} = g$. Set $u = \bar{g} + w$ for some $w \in H_0^1(\Omega)$. Then the trace of u is $g + 0 = g$. If $Lu = f$ in Ω then

$$0 = Lu - f = Lw + L\bar{g} - f = Lw - \tilde{f},$$

where $\tilde{f} = f - L\bar{g}$. Hence, w should be a weak solution of $Lw = \tilde{f}$ with zero boundary data.

(b) Continuing from above, the regularity of $u = \bar{g} + w$ is governed by the regularity of $\bar{g}$ and of w . The coefficients in L are smooth, so the solution of $Lw = \tilde{f}$ has regularity H_{loc}^{m+2} whenever $\tilde{f} \in H^m(\Omega)$. To this end, we need both f and $L\bar{g}$ to lie in H^m . The function f given in (1) is smooth, so it lies in H^m for all m . Since L takes two derivatives, we would need $\bar{g} \in H^{m+2}$ to have $L\bar{g} \in H^m$.

To summarize, $u \in H_{\text{loc}}^{m+2}(\Omega)$ provided $\bar{g} \in H^{m+2}(\Omega)$.

- (i) A natural extension is $\bar{g}(x) = e^{x_2 - x_1}$, which is smooth, and hence lies in H^m for all m . Hence, the solution u lies in $H_{\text{loc}}^m(\Omega)$ for all m .
- (ii) A natural extension would be $\bar{g}(x) = |x - x_0|^{7/3}$. We have seen that functions of the form $|x|^\beta$ for $x \in \mathbb{R}^n$ lie in $W^{k,p}$ provided $p(\beta - k) + n > 0$, that is, $k < \beta + n/p$. We have $p = 2$, $\beta = 7/3$ and $n = 3$, so we get $k < 7/3 + 3/2 = 23/6 = 3.8333 \dots$. Therefore we can pick $k = 3$, but not higher. From the above discussion we get $u \in H_{\text{loc}}^3(\Omega)$, but not more.

4. Let $\Omega \subset \mathbb{R}^3$ be open and bounded and let $f \in H^2(\Omega)$. Show that any weak solution $u \in H^1(\Omega)$ of

$$-\Delta u = f \quad \text{in } \Omega \tag{4}$$

is also a classical solution.

Solution: The coefficients of $L = -\Delta$ are smooth, and $f \in H^2(\Omega)$, so according to the interior regularity theory, any weak solution u of (4) will lie in $H_{\text{loc}}^4(\Omega)$, that is, $u \in H^4(V)$ for any $V \Subset \Omega$. According to Morrey's inequality,

$$W^{k,p}(V) \hookrightarrow C^{m,\gamma}(V), \quad m = k - 1 - \left\lfloor \frac{n}{p} \right\rfloor, \quad \gamma = 1 - \left(\frac{n}{p} - \left\lfloor \frac{n}{p} \right\rfloor \right)$$

(as long as $n/p \notin \mathbb{Z}$). Setting $k = 4$, $p = 2$, $n = 3$ gives $m = 4 - 1 - 1 = 2$, $\gamma = 1/2$, so $u \in C^{2,1/2}(V)$ for any set $V \Subset \Omega$. In particular, u is C^2 everywhere in Ω . Hence, u satisfies (4) pointwise — it is a classical solution.