

Problem sheet 6

MAT4305

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You will find a collection of hints on page 3.

1. Let $\Omega \subset \mathbb{R}^n$ be open, bounded and connected, let $p \in [1, \infty]$ and let $u \in W^{1,p}(\Omega)$ satisfy $Du = 0$ in Ω . Show that u is constant.
2. (Problem 6.6.5 in Evans) Let $\Omega \subset \mathbb{R}^n$ be an open, bounded set with C^1 boundary. Consider the Poisson equation with *Robin* boundary conditions,

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u + \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases} \quad (1)$$

where $f \in L^2(\Omega)$ and $\frac{\partial u}{\partial \nu} = \nu \cdot Du$ is the normal derivative on $\partial\Omega$.

- (a) Define what it means to be a weak solution of (1).
- (b) Show that your bilinear form $B[u, v]$ is well-defined and continuous.
- (c) Use Problem 1 directly to show that there exists at most one weak solution of (1).
- (d) Show that B is coercive on $H^1(\Omega)$, that is, there is a $\beta > 0$ such that

$$B[u, u] \geq \beta \|u\|_{H^1}^2 \quad \forall u \in H^1(\Omega). \quad (2)$$

- (e) Conclude that (1) has a unique weak solution, and that the solution satisfies

$$\|u\|_{H^1} \leq C \|f\|_{L^2}$$

for some $C > 0$ independent of f .

3. Let $\Omega \subset \mathbb{R}^n$ be open and bounded. Consider the boundary-value problem

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad (3)$$

for an elliptic operator L and some $f \in L^2(\Omega)$. We say that $\lambda \in \mathbb{R}$ is an eigenvalue of this boundary-value problem if there exists a nonzero weak solution $w \in H_0^1(\Omega)$ of the equation $Lw = \lambda w$.

- (a) Explain why the above definition of an eigenvalue is equivalent to the following: A number $\lambda \in \mathbb{R}$ is *not* an eigenvalue of (3) if and only if the problem

$$\begin{cases} Lu = \lambda u + f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad (4)$$

has a unique weak solution $u \in H_0^1(\Omega)$, for any $f \in L^2(\Omega)$.

- (b) Let now $n = 2$, $\Omega = (0, 1) \times (0, 1)$, $L = -\Delta$. Show that

$$w_k(x) := \sin(k_1\pi x_1) \sin(k_2\pi x_2), \quad k = (k_1, k_2) \in \mathbb{N}^2$$

are the eigenvectors of (3). Find the corresponding eigenvalues.

- (c) Continuing from problem (b), consider the heat equation

$$\begin{cases} \partial_t u = \Delta u & \text{for } x \in \Omega, t > 0, \\ u = 0 & \text{on } \partial\Omega \times (0, \infty), \\ u = g & \text{on } \Omega \times \{0\}. \end{cases} \quad (5)$$

Solve (5) when g is one of the eigenvectors from problem (b). Then solve (5) when g is a linear combination of eigenvectors. Finally, solve (5) for an arbitrary $g \in L^2(\Omega)$.

- (d) What is the long-term behaviour of solutions of (5)?
- (e) It is a fact from Fourier analysis that the eigenvectors from (b) constitute a basis for $L^2(\Omega)$, and that if $h \in L^2(\Omega)$ has basis expansion $h(x) = \sum_{k \in \mathbb{N}^2} \alpha_k w_k$, where $|\alpha_k| \leq C|k|^{-r-2}$ for all $k \in \mathbb{N}^2$ for some $C > 0$ and $r \in \mathbb{N}$, then $f \in C^r(\Omega)$. Use this to prove that $u(\cdot, t) \in C^\infty(\Omega)$ for all $t > 0$.

Hints

2. (c)
- Assume conversely that no such $\beta > 0$ exists; then there are $u_1, u_2, \dots \in H^1(\Omega)$ with $\|u_k\|_{H^1} = 1$ but $B[u_k, u_k] \rightarrow 0$. Show that there is a subsequence which converges weakly in H^1 and strongly in L^2 .
 - Show that the subsequence actually converges strongly in H^1 .
 - Reach a contradiction.