

Problem sheet 7 – Solutions

MAT4305

Ulrik Skre Fjordholm

June 5, 2023

1. Let $\Omega \subset \mathbb{R}^n$ be open, bounded and connected, let $p \in [1, \infty]$ and let $u \in W^{1,p}(\Omega)$ satisfy $Du = 0$ in Ω . Show that u is constant.

Solution: Assume first that $u \in C^1(\Omega)$. If $B(x, r) \subset \Omega$ then for any $y \in B(x, r)$,

$$u(y) - u(x) = \int_0^1 \frac{d}{ds} u(x + s(y-x)) ds = \int_0^1 Du(x + s(y-x)) \cdot (y-x) ds = 0,$$

so u is constant in $B(x, r)$. Fixing any $x_0 \in \Omega$, the above shows that the set

$$A := \{x \in \Omega : u(x) = u(x_0)\}$$

is an open subset of Ω . Since u is continuous, the set

$$\Omega \setminus A = \{x \in \Omega : u(x) \neq u(x_0)\} = u^{-1}(\mathbb{R} \setminus \{u(x_0)\})$$

is also open (being the continuous inverse of an open set). But Ω is connected, so it cannot be written as the disjoint union of two open, nonempty sets, whence $\Omega \setminus A = \emptyset$, and so $u \equiv u(x_0)$.

If only $u \in W^{1,p}(\Omega)$ then in particular $u \in W^{1,1}(\Omega)$. If $u_\varepsilon := \eta_\varepsilon * u$ then $u_\varepsilon \in C^1(\Omega_\varepsilon)$ for $x \in \Omega_\varepsilon := \{x \in \Omega : \text{dist}(x, \partial\Omega) > \varepsilon\}$. Moreover, $Du_\varepsilon = 0$ in Ω_ε , so u_ε is constant in each connected component of Ω_ε . We know by the First Approximation Theorem that $u_\varepsilon \rightarrow u$ in $W_{\text{loc}}^{1,1}(\Omega)$, so in particular, u is constant on each connected component of Ω_ε , for any $\varepsilon > 0$. Letting $\varepsilon \rightarrow 0$, the components of Ω_ε “join together”, and we conclude that u is equal to one fixed constant on all of Ω .

2. (Problem 6.6.5 in Evans) Let $\Omega \subset \mathbb{R}^n$ be an open, bounded set with C^1 boundary. Consider the Poisson equation with *Robin* boundary conditions,

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u + \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases} \quad (1)$$

where $f \in L^2(\Omega)$ and $\frac{\partial u}{\partial \nu} = \nu \cdot Du$ is the normal derivative on $\partial\Omega$.

- (a) Define what it means to be a weak solution of (1).
- (b) Show that your bilinear form $B[u, v]$ is well-defined and continuous.
- (c) Use Problem 1 directly to show that there exists at most one weak solution of (1).

(d) Show that B is coercive on $H^1(\Omega)$, that is, there is a $\beta > 0$ such that

$$B[u, u] \geq \beta \|u\|_{H^1}^2 \quad \forall u \in H^1(\Omega). \quad (2)$$

(e) Conclude that (1) has a unique weak solution, and that the solution satisfies

$$\|u\|_{H^1} \leq C \|f\|_{L^2}$$

for some $C > 0$ independent of f .

Solution:

(a) Multiply the PDE by a test function v and integrate over Ω :

$$\begin{aligned} (f, v)_{L^2(\Omega)} &= - \int_{\Omega} v \Delta u \, dx = - \int_{\partial\Omega} v \frac{\partial u}{\partial \nu} dS(x) + \int_{\Omega} Dv \cdot Du \\ &= \int_{\partial\Omega} uv \, dS(x) + \int_{\Omega} Du \cdot Dv \, dx =: B[u, v]. \end{aligned}$$

This leads us to the following definition: A function $u \in H^1(\Omega)$ is a weak solution of (1) if $B[u, v] = (f, v)_{L^2}$ for all $v \in H^1(\Omega)$.

(b) The bilinear form is well-defined if we interpret u, v on $\partial\Omega$ as the trace of u and v . If $C > 0$ is such that $\|Tu\|_{L^2(\partial\Omega)} \leq C \|u\|_{H^1(\Omega)}$ then

$$\begin{aligned} |B[u, v]| &\leq \|u\|_{L^2(\partial\Omega)} \|v\|_{L^2(\partial\Omega)} + \|Du\|_{L^2(\Omega)} \|Dv\|_{L^2(\Omega)} \\ &\leq C^2 \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)} + \|Du\|_{L^2(\Omega)} \|Dv\|_{L^2(\Omega)} \\ &\leq (1 + C^2) \|u\|_{H^1} \|v\|_{H^1} \end{aligned}$$

so B is continuous.

(c) Let u_1, u_2 be two solutions of (1) and let $u = u_1 - u_2$. Then u solves (1) with $f = 0$. Hence, $B[u, v] = 0$ for all $v \in H^1(\Omega)$. Setting $v = u$ gives

$$0 = B[u, u] = \int_{\partial\Omega} u^2 \, dS(x) + \int_{\Omega} |Du|^2 \, dx$$

so $u = 0$ on $\partial\Omega$ and $Du = 0$ in Ω . By problem 1 we see that $u \equiv c$ for some constant $c \in \mathbb{R}$, and since $u = 0$ on $\partial\Omega$, we conclude $c = 0$. In other words, $u_1 = u_2$.

(d) Assume that there is no such $\beta > 0$. Then there exist $u_1, u_2, \dots \in H^1(\Omega)$ such that $\|u_k\|_{H^1} = 1$ for all k , but $B[u_k, u_k] \rightarrow 0$ as $k \rightarrow \infty$. Since $(u_k)_{k \in \mathbb{N}}$ is bounded in the Hilbert space $H^1(\Omega)$, the sequence has a weakly convergent subsequence. Moreover, we have $H^1(\Omega) \subseteq L^2(\Omega)$, so this subsequence has yet another subsequence which converges strongly in $L^2(\Omega)$. Denoting the subsequence by $(u_{k(m)})_{m \in \mathbb{N}}$, we have

$$u_{k(m)} \rightarrow u \quad \text{in } L^2(\Omega), \quad u_{k(m)} \rightharpoonup u \quad \text{in } H^1(\Omega)$$

as $m \rightarrow \infty$. On the other hand, we have

$$0 = \lim_{m \rightarrow \infty} B[u_{k(m)}, u_{k(m)}] = \lim_{m \rightarrow \infty} (\|Du_{k(m)}\|_{L^2(\Omega)}^2 + \|u_{k(m)}\|_{L^2(\partial\Omega)}^2),$$

so in fact $Du_{k(m)} \rightarrow 0$ in $L^2(\Omega)$. But in light of the convergence of $(u_{k(m)})_m$, the subsequence must actually converge strongly in H^1 , and the limit u satisfies $Du = \lim_m Du_{k(m)} = 0$. From the above we can also deduce that $Tu = \lim_{m \rightarrow \infty} Tu_{k(m)} = 0$. Hence, applying problem 1, we conclude that $u = 0$. But then

$$1 = \|u_{k(m)}\|_{H^1(\Omega)} \rightarrow \|u\|_{H^1(\Omega)} = 0,$$

a contradiction.

- (e) Existence and uniqueness now follows from Lax–Milgram. Letting u denote this solution, we get

$$\beta \|u\|_{H^1}^2 \leq B[u, u] = (f, u)_{L^2} \leq \|f\|_{L^2} \|u\|_{L^2} \leq \|f\|_{L^2} \|u\|_{H^1}.$$

Dividing by $\beta \|u\|_{H^1}$ now yields the estimate with $C = \beta^{-1}$.

3. Let $\Omega \subset \mathbb{R}^n$ be open and bounded. Consider the boundary-value problem

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad (3)$$

for an elliptic operator L and some $f \in L^2(\Omega)$. We say that $\lambda \in \mathbb{R}$ is an eigenvalue of this boundary-value problem if there exists a nonzero weak solution $w \in H_0^1(\Omega)$ of the equation $Lw = \lambda w$.

- (a) Explain why the above definition of an eigenvalue is equivalent to the following: A number $\lambda \in \mathbb{R}$ is *not* an eigenvalue of (3) if and only if the problem

$$\begin{cases} Lu = \lambda u + f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad (4)$$

has a unique weak solution $u \in H_0^1(\Omega)$, for any $f \in L^2(\Omega)$.

- (b) Let now $n = 2$, $\Omega = (0, 1) \times (0, 1)$, $L = -\Delta$. Show that

$$w_k(x) := \sin(k_1 \pi x_1) \sin(k_2 \pi x_2), \quad k = (k_1, k_2) \in \mathbb{N}^2$$

are the eigenvectors of (3). Find the corresponding eigenvalues.

- (c) Continuing from problem (b), consider the heat equation

$$\begin{cases} \partial_t u = \Delta u & \text{for } x \in \Omega, t > 0, \\ u = 0 & \text{on } \partial\Omega \times (0, \infty), \\ u = g & \text{on } \Omega \times \{0\}. \end{cases} \quad (5)$$

Solve (5) when g is one of the eigenvectors from problem (b). Then solve (5) when g is a linear combination of eigenvectors. Finally, solve (5) for an arbitrary $g \in L^2(\Omega)$.

- (d) What is the long-term behaviour of solutions of (5)?

- (e) It is a fact from Fourier analysis that the eigenvectors from (b) constitute a basis for $L^2(\Omega)$, and that if $h \in L^2(\Omega)$ has basis expansion $h(x) = \sum_{k \in \mathbb{N}^2} \alpha_k w_k$, where $|\alpha_k| \leq C |k|^{-r-2}$ for all $k \in \mathbb{N}^2$ for some $C > 0$ and $r \in \mathbb{N}$, then $f \in C^r(\Omega)$. Use this to prove that $u(\cdot, t) \in C^\infty(\Omega)$ for all $t > 0$.

Solution:

- (a) Let Σ_1 be the set of $\lambda \in \mathbb{R}$ satisfying the first definition, and Σ_2 the set of those satisfying the second. Applying the Second Existence Theorem to (4) for an arbitrary $\lambda \in \mathbb{R}$, we see that *either* (4) has a unique solution for all $f \in L^2(\Omega)$ (that is, $\lambda \in \Sigma_2^c$), *or* the equation $Lu = \lambda u$ has a nonzero solution (that is, $\lambda \in \Sigma_1$). Hence, $\Sigma_1 = (\Sigma_2^c)^c = \Sigma_2$, so the two definitions coincide.
- (b) Set $w_k(x) = \sin(k_1 \pi x_1) \sin(k_2 \pi x_2)$ for $k \in \mathbb{N}^2$; these functions clearly lie in $H_0^1(\Omega)$. (If $k_1 = 0$ or $k_2 = 0$ then $w_k = 0$, while if $k_1 < 0$ (or $k_2 < 0$) then $w_k = w_{(-k_1, k_2)}$, so it is enough to consider $k \in \mathbb{N}^2$.) Then

$$-\Delta w_k(x) = (k_1 \pi)^2 w_k(x) + (k_2 \pi)^2 w_k(x) = \lambda_k w_k(x)$$

(in the classical sense) where $\lambda_k := |k|^2 \pi^2 = (k_1 \pi)^2 + (k_2 \pi)^2$. These functions are orthogonal in $L^2(\Omega)$ and span all of $L^2(\Omega)$, so by a theorem in functional analysis, there are no more eigenpairs.

- (c) If $g = w_k$ we set as an ansatz $u(x, t) = c(t)w_k(x)$. Then

$$\partial_t u - \Delta u = c'(t)w_k + \lambda_k w_k,$$

so we need $c(0) = 1$ and $c'(t) = -\lambda_k$, that is, $c(t) = e^{-\lambda_k t}$. Thus,

$$u(x, t) = e^{-|k|^2 \pi^2 t} \sin(k_1 \pi x_1) \sin(k_2 \pi x_2).$$

Taking linear combinations of such initial data leads to corresponding linear combinations of the solutions:

$$\begin{aligned} g &= \sum_{k, \ell=0}^N \alpha_k w_k \\ \Rightarrow u(x, t) &= \sum_{k, \ell=0}^N \alpha_k e^{-|k|^2 \pi^2 t} \sin(k_1 \pi x_1) \sin(k_2 \pi x_2). \end{aligned}$$

Finally, by a result in Fourier analysis, the functions w_k constitute a basis for $L^2(\Omega)$, so we can write any $g \in L^2(\Omega)$ as the convergent series

$$g = \sum_{k \in \mathbb{N}^2} \alpha_k w_k.$$

We get the resulting solution

$$u(x, t) = \sum_{k \in \mathbb{N}^2} \alpha_k e^{-|k|^2 \pi^2 t} \sin(k_1 \pi x_1) \sin(k_2 \pi x_2).$$

- (d) Noting that $e^{-|k|^2\pi^2 t} \rightarrow 0$ as $t \rightarrow \infty$, and that the convergence happens faster for larger $|k|$, we see that all oscillations are quickly dampened. The solution converges to 0 as $t \rightarrow \infty$.
- (e) With the above basis expansion of $u(\cdot, t)$, we see that the Fourier coefficients converge to 0 faster than any polynomial as $|k| \rightarrow \infty$. Therefore, $u(\cdot, t) \in C^r(\Omega)$ for any $r \in \mathbb{N}$.

Hints

2. (c)
- Assume conversely that no such $\beta > 0$ exists; then there are $u_1, u_2, \dots \in H^1(\Omega)$ with $\|u_k\|_{H^1} = 1$ but $B[u_k, u_k] \rightarrow 0$. Show that there is a subsequence which converges weakly in H^1 and strongly in L^2 .
 - Show that the subsequence actually converges strongly in H^1 .
 - Reach a contradiction.