Mandatory assignment for MAT4360: C*-algebras (corrected version) Høst 2016/17
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The solution to the assignment must be delivered in the special box ('obligkassa') on the 7th floor in the Niels Henrik Abel building before 14:30 on Thursday, 13 October, 2016. There are signs pointing to the box when you come out of the elevator on the 7th floor. Remember to fill in a cover page for the assignment before you put it into the box. Blank cover pages can be found near the 'obligkassa' or here. You are strongly encouraged to deliver the solution typeset in LaTeX.
You must justify all your answers. If a proof is longer, a brief description of the rough strategy at the start will make it easier for the reader (and yourself) to follow.
Please read the general information on mandatory assignments available in English and Norwegian.

Task 1:

(a) Let A be a unital C^*-algebra. Show that every element in A is the linear combination of four unitaries in A [Hint: Consider $a \pm i\sqrt{1-a^2}$ for a self-adjoint a with $\|a\| \leq 1$.].

(b) Let H be a Hilbert space and A be a C^*-subalgebra of $\mathcal{L}(H)$ with $1 \in A$. Show that $T \in \mathcal{L}(H)$ belongs to A'' if and only if $uTu^* = T$ for all unitaries u in A'.

Task 2:
Let A be a C^*-algebra. A *right-Hilbert module* over A is a Banach space E with a bilinear map $E \times A \to E, (\xi, a) \mapsto \xi a$, and a sesquilinear map $\langle \cdot, \cdot \rangle \colon E \times E \to A$ such that

$$\langle \xi, \eta \rangle^* = \langle \eta, \xi \rangle, \quad \langle \xi, \eta a \rangle = \langle \xi, \eta \rangle a, \quad \|\langle \xi, \xi \rangle\| = \|\xi\|^2, \quad \text{and } \langle \xi, \xi \rangle \geq 0$$

for all $\xi, \eta \in E$ and $a \in A$. In every right-Hilbert module E, the Cauchy-Schwarz inequality $\|\langle \xi, \eta \rangle\|^2 \leq \|\xi\|^2 \|\eta\|^2$ holds for all $\xi, \eta \in E$ (we cannot prove this based on what we know already, so we take it for granted here).
Given two right-Hilbert modules E and F over A, a map $S \colon E \to F$ is *adjointable* if there is a map $T \colon F \to E$ with $\langle S\xi, \eta \rangle_F = \langle \xi, T\eta \rangle_E$ for all $\xi \in E, \eta \in F$. If such a T exists, it is uniquely determined. It is called the *adjoint* of S, and is denoted by S^*. The collection of all adjointable linear maps $S \colon E \to F$ is denoted by $\mathcal{L}_A(E, F)$, and we set $\mathcal{L}_A(E) := \mathcal{L}_A(E, E)$.

(a) Show that E is a right module over the algebra A, that is, $(\xi a)b = \xi(ab)$ for all $a, b \in A, \xi \in E$.

(b) Show that $(\xi u_\lambda)_{\lambda \in \Lambda}$ converges to ξ for every $\xi \in E$ and every approximate unit $(u_\lambda)_{\lambda \in \Lambda}$ in A.

(c) Show that $S^{**} = S$ for every $S \in \mathcal{L}_A(E, F)$ and conclude that $\mathcal{L}_A(E)$ is a C^*-algebra for the operator norm on E and involution $S \mapsto S^*$.

(d) Show that $\mathcal{L}_A(E, F)$ is a right-Hilbert module over $\mathcal{L}_A(E)$.

(e) Show that for all $\xi \in E, \eta \in F$, the map $|\eta\rangle\langle\xi| \colon E \to F, \zeta \mapsto \eta\langle \xi, \zeta \rangle_E$ is adjointable and determine its adjoint.

(f) Let $\mathcal{K}_A(E, F) := \overline{\operatorname{span}\{|\eta\rangle\langle\xi| \mid \xi \in E, \eta \in F\}}$. Show that $\mathcal{L}_A(F)\mathcal{K}_A(E, F)\mathcal{L}_A(E) = \mathcal{K}_A(E, F)$, and deduce that $\mathcal{K}_A(E) := \mathcal{K}_A(E, E)$ is a closed two-sided ideal in $\mathcal{L}_A(E)$.

(g) Show that every closed two-sided ideal I in a C^*-algebra A has a canonical structure of a right-Hilbert module over A.

Task 3:
Let H be a Hilbert space. Denote by $\mathcal{L}(H)_{sa}$ the set of self-adjoint, bounded, linear operators on H, and by $\mathcal{U}(H)$ the group of unitaries in $\mathcal{L}(H)$. Show that the assignment $T \mapsto U_T := (T+i)(T-i)^{-1}$ defines a strongly continuous bijection between $\mathcal{L}(H)_{sa}$ and $\{U \in \mathcal{U}(H) \mid 1 \notin \operatorname{Sp} U\}$ by proving the following steps:

(a) Show that $U_T \in \mathcal{U}(H)$ for every $T \in \mathcal{L}(H)_{sa}$.

(b) Show that for $U \in \mathcal{U}(H)$ with $1 \notin \operatorname{Sp} U$, the operator $T_U := i(U+1)(U-1)^{-1}$ is self-adjoint, and satisfies $T_{U_T} = T, U_{T_U} = U$.

(c) Show that $(S+i)(U_S - U_T)(T+i) = 2i(T-S)$ for $S, T \in \mathcal{L}(H)_{sa}$, and use this to show that the map $T \mapsto U_T$ is strongly continuous on $\mathcal{L}(H)_{sa}$.

Task 4:
Show that every $f \in C_0(\mathbb{R}, \mathbb{R})$ defines a strongly continuous map $\mathcal{L}(H)_{sa} \to \mathcal{L}(H)_{sa}, T \mapsto f(T)$.
[Hint: Consider $g\colon \mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\} \to \mathbb{R}$ given by $g(1) := 0$ and $g(z) := f(i(z+1)(z-1)^{-1})$ for $z \neq 1$. Show that $g \in C(\mathbb{T}, \mathbb{R})$ and $g(U_T) = f(T)$ with U_T from Task 3. Then use (without proof) that every continuous $\mathbb{C}$-valued function is strongly continuous on bounded sets of normal operators in $\mathcal{L}(H)$.]

Task 5:
Suppose A is a $*$-subalgebra of $\mathcal{L}(H)$ for some Hilbert space H. Let $A_1 := \{a \in A \mid \|a\| \leq 1\}$, and for $M \subset \mathcal{L}(H)$ denote by $\overline{M}$ the closure of M with respect to the strong operator topology. Show that

(i) $A_{sa,1}$ is strongly dense in $(\overline{A})_{sa,1}$,

(ii) A_1 is strongly dense in $(\overline{A})_1$, and

(iii) $A_{+,1}$ is strongly dense in $(\overline{A})_{+,1}$.

To do so, establish the following intermediate steps:

(a) Every $T \in (\overline{A})_{sa}$ can be strongly approximated by a net in A_{sa}.
[Hint: A_{sa} is convex, and you can assume without proof that weak closure and strong closure coincide on convex subsets of $\mathcal{L}(H)$.]

(b) Produce a function $f \in C_0(\mathbb{R}, \mathbb{R})$ with $f|_{[-1,1]} = \mathrm{id}$. Then use (a) in combination with Task 3 and Task 4 to prove (i).

(c) Prove (ii) from (i) by considering $\begin{pmatrix} 0 & T^* \\ T & 0 \end{pmatrix} \in \mathcal{L}(H \oplus H)$ for $T \in (\overline{A})_1$.

(d) Prove (iii) from (i) by first showing that $T \in (\overline{A})_{+,1}$ implies $T = S^*S$ for some $S \in (\overline{A})_{+,1}$.