UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in MAT3400/4400 — Linear analysis with applications.

Day of examination: Monday, December 6, 2010.

Examination hours: 9.00 – 13.00.

This problem set consists of 2 pages.

Appendices: None.

Permitted aids: None.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Note: You must write proofs for all your answers!

Problem 1

Let $L^2[-\pi,\pi]$ be the Hilbert space of square-integrable functions on $[-\pi,\pi]$ with respect to Lebesgue measure where the inner product is given by $\langle f,g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi}\overline{f(x)}g(x)dx$. It is assumed known that the functions $e_n(x) = e^{inx}$ for $n \in \mathbb{Z}$ form an orthonormal basis of $L^2[-\pi,\pi]$.

1a

Compute the Fourier series of the function $f(x) = x^2$ on $[-\pi,\pi]$. Express the answer in terms of trigonometric functions.

1b

Consider the Fourier series in question (a): Is it pointwise convergent? Is it uniformly convergent?

1c

Use question (b) to find the sum of the series $\sum_{m=1}^{\infty}(-1)^{m+1}\frac{1}{m^2}$.

Problem 2

Let $a < b$ be real numbers, and let κ be a continuous function on $[a,b]\times[a,b]$ with values in $\mathbb{C}$ such that $\kappa(x,y) = \overline{\kappa(y,x)}$ for all $x,y \in [a,b]$. Let H be the Hilbert space $L^2[a,b]$ of square-integrable functions on $[a,b]$ with respect to

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Lebesgue measure where the inner product is given by $\langle f,g\rangle = \int_a^b \overline{f(y)}g(y)dy$. Let K denote the compact self-adjoint operator on H given by

$$(Kf)(x) = \int_a^b \kappa(x,y)f(y)dy.$$

2a

Assume $\{\lambda_j\}_{j=1}^{\infty}$ are the eigenvalues of K, and let $\{e_j\}_{j=1}^{\infty}$ be an orthonormal sequence in $L^2[a,b]$ where e_j is an eigenvector corresponding to λ_j for every $j \geq 1$. Show that for every $x \in [a,b]$ we have $\sum_{j=1}^{\infty} |(Ke_j)(x)|^2 \leq M^2(b-a)$, where we let $M = \sup\{|\kappa(x,y)| \mid x,y \in [a,b]\}$. (Hint: Use Bessel's inequality for the function $y \mapsto \kappa(y,x)$.)

2b

Use the monotone convergence theorem to show that $\sum_{j=1}^{\infty} \lambda_j^2 < M^2(b-a)^2$.

Problem 3

Let $L^2[0,1]$ be the Hilbert space of square-integrable functions on $[0,1]$ with respect to Lebesgue measure where the inner product is $\langle f,g\rangle = \int_0^1 \overline{f(y)}g(y)dy$. Consider the following Sturm-Liouville operator: $Lu = -u''$ with domain $\mathcal{D}(L) = \{f \in C^2[0,1] \mid f(0) = 0, f'(1) = 0\} \subset L^2[0,1]$. It is assumed known that all eigenvalues of L are real.

Show that $\alpha_n = (n - \frac{1}{2})^2\pi^2$, $n = 1,2,\ldots$ are the eigenvalues of L. Find corresponding normalized eigenvectors $\{u_n\}_{n\geq 1}$. Is L injective?

Problem 4

Let H be an infinite-dimensional Hilbert space and K a compact, self-adjoint operator on H such that $\ker(K) = \{0\}$. Find a sequence $\{A_n\}_{n\geq 1}$ of finite rank operators on H such that for each x in H we have

$$\lim_{n\to\infty} A_nKx = x \text{ and } \lim_{n\to\infty} KA_nx = x.$$

END.