UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in	MAT3400/4400 — Linear analysis with applications.
Day of examination:	Monday, December 5, 2011.
Examination hours:	9.00 – 13.00.

This problem set consists of 3 pages.

Appendices:	None.
Permitted aids:	None.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Note: You must justify all your answers!

Problem 1

(20 points) Let $L^2([-\pi,\pi])$ be the Hilbert space of equivalence classes of square-integrable functions on $[-\pi,\pi]$ with respect to Lebesgue measure. The inner product on $L^2([-\pi,\pi])$ is given by $\langle f,g\rangle = \int_{-\pi}^{\pi} \overline{f(x)}g(x)dx$ for $f,g \in \mathcal{L}^2([-\pi,\pi])$. It is assumed known that the functions $e_n(x) = \frac{1}{\sqrt{2\pi}}e^{inx}$ for $n \in \mathbb{Z}$ form an orthonormal basis of $L^2([-\pi,\pi])$.

Compute the Fourier series of the function

$$f(x) = \begin{cases} -1 & \text{if } -\pi < x < 0 \\ 0 & \text{if } 0 \le x < \pi. \end{cases}$$

Express the answer in terms of trigonometric functions. You can use the fact that $\sin t = \frac{1}{2i}(e^{it} - e^{-it})$ for all $t \in \mathbb{R}$.

Problem 2

Let $\kappa(x,y) = \frac{1}{3} - xy$ be defined on $[0,1]\times[0,1]$. Let H be the Hilbert space $L^2([0,1])$ of equivalence classes of square-integrable functions on $[0,1]$ with respect to Lebesgue measure where the inner product is given by $\langle f,g\rangle = \int_0^1 \overline{f(x)}g(x)dx$ for $f,g \in \mathcal{L}^2([0,1])$. Let K denote the compact self-adjoint integral operator on H given by

$$(Kf)(x) = \int_0^1 \kappa(x,y)f(y)dy.$$

2a

(10 points) Find the eigenvalues and corresponding eigenvectors for K.

(Continued on page 2.)

2b

(10 points) Formulate the Fredholm alternative for a compact operator on a Hilbert space. In the case of K from question 2a decide for which $g \in H$ there is a unique solution for the equation $f = Kf + g$.

Problem 3

Let $L^2([0,1])$ be the Hilbert space of equivalence classes of square-integrable functions on $[0,1]$ with respect to Lebesgue measure where the inner product is $\langle f, g\rangle = \int_0^1 \overline{f(x)} g(x) dx$ for $f, g \in \mathcal{L}^2([0,1])$. It is known that $C^2([0,1]) \subset L^2([0,1])$.

3a

(10 points) Let $q : [0,1] \to [0,\infty)$ be a continuous function and let $Lu = -u'' + qu$ be the Sturm-Liouville operator on a domain $\mathcal{D}(L) \subset C^2([0,1])$. It is assumed known that L can only have real eigenvalues. Suppose that $u(1)u'(1) - u(0)u'(0) \leq 0$ for all $u \in \mathcal{D}(L)$ such that $Lu = \alpha u$. Prove that

$$\alpha \int_0^1 u^2(x)dx \geq \int_0^1 q(x)u^2(x)dx + \int_0^1 (u'(x))^2 dx.$$

Conclude that all eigenvalues of L are positive or zero.

3b

(20 points) Let L be given by $Lu = -u''$ on $\mathcal{D}(L) = \{u \in C^2([0,1]) : u'(0) = 0, u'(1) = 0\}$. Find the eigenvalues and corresponding normalized eigenvectors for L.

Problem 4

Let (X, Σ, μ) be a measure space and let λ be Lebesgue measure on the Borel σ-algebra $\mathcal{B}$ on $\mathbb{R}$. Let $1 \leq p < \infty$.

4a

(5 points) Let $g : (0,\infty) \to \mathbb{R}$ be the function $g(t) = t^{p-1}$. Prove that $\int_{(0,a]} g d\lambda = \frac{1}{p} a^p$ for every $a > 0$.

Let $f : X \to [0,\infty]$ be a measurable function. Define a function $\phi_f : (0,\infty) \to [0,\infty]$ by $\phi_f(t) = \mu(\{x \in X : f(x) > t\})$.

4b

(5 points) Suppose that f has form $f = a\chi_A$ where $a \geq 0$ and $A \in \Sigma$. Prove that $\phi_f = \mu(A)\chi_{(0,a)}$ and $a^p \mu(A) = p \int_{(0,\infty)} g \phi_f d\lambda$.

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4c

(10 points) Suppose that f has form $f = \sum_{j=1}^{m} f_j$ where $f_j = a_j \chi_{A_j}$ with $a_j \geq 0$ and $A_j \in \Sigma$ for all $j = 1, \ldots, m$, and such that $A_j \cap A_k = \emptyset$ if $j \neq k$ (thus f is a simple, measurable, nonnegative function in standard form). Prove that ϕ_f is Borel measurable and that

$$\int_X f^p d\mu = p \int_{(0,\infty)} g \phi_f d\lambda. \tag{1}$$

Hint: show that $\phi_f = \sum_{j=1}^{m} \phi_{f_j}$.

4d

(10 points) For arbitrary $f : X \to [0, \infty]$ measurable prove that ϕ_f is Borel measurable and that the equality (1) is valid.

END