UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in: MAT3400/4400 — Linear analysis with applications

Day of examination: Thursday, December 6, 2012

Examination hours: 09.00 – 13.00

This problem set consists of 2 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1

Using Fourier series show that if $f\colon [0, 2\pi] \to \mathbb{C}$ is a C^1-function such that $f(0) = f(2\pi)$ and $\int_0^{2\pi} f(t)dt = 0$, then

$$\int_0^{2\pi} |f(t)|^2 dt \le \int_0^{2\pi} |f'(t)|^2 dt.$$

Describe all functions f as above such that the equality holds.

Problem 2

Assume $(a_{ij})_{i,j=1}^\infty$ is an infinite matrix with complex coefficients such that $\sum_{i,j=1}^\infty |a_{ij}|^2 < \infty$.

2a

Consider the Hilbert space ℓ_2. Show that the following formula defines a bounded linear operator T from ℓ_2 into itself:

$$T(x_1, x_2, \dots) = \left(\sum_{j=1}^\infty a_{1j} x_j, \sum_{j=1}^\infty a_{2j} x_j, \dots \right).$$

2b

Show that T is a Hilbert-Schmidt operator and compute its Hilbert-Schmidt norm.

(Continued on page 2.)

Problem 3

Consider the operator $T: L^2[0,1] \to L^2[0,1]$ defined by

$$(Tf)(s) = \int_0^s f(t)dt, \quad s \in [0,1].$$

It is assumed to be known that T is a compact operator and that the image of T is contained in the subspace $L^2_{cont}[0,1] \subset L^2[0,1]$ of continuous functions.

3a

Show that the adjoint operator is given by

$$(T^*f)(s) = \int_s^1 f(t)dt, \quad s \in [0,1].$$

3b

Show that a function $f \in L^2[0,1]$ is an eigenvector of T^*T with eigenvalue $\lambda > 0$ if and only if it is smooth (more pedantically, it coincides a.e. with a smooth function) and satisfies the differential equation

$$\begin{cases} \lambda f'' + f = 0, \\ f(1) = 0, \ f'(0) = 0. \end{cases}$$

3c

Find the singular values of T (it can be shown that the kernel of T is zero, but you don't have to prove this). What is the operator norm of T? Is T a trace-class operator?

Problem 4

4a

Assume that the Fourier coefficients of a function $f \in L^1[0, 2\pi]$ are all zero. Show that $f = 0$ a.e.

(There are several ways of proving this. One proof relies on the following result, which was one of the exercises in the course and is therefore assumed to be known: the space of bounded Borel functions on $[0, 2\pi]$ is the smallest space of functions that contains $C[0, 2\pi]$ and that is closed under pointwise limits of bounded sequences of functions.)

4b

Assume $f \in L^1[0, 2\pi]$ is such that its Fourier coefficients $c_n(f)$ satisfy $\sum_{n\in\mathbb{Z}} |c_n(f)|^2 < \infty$. Show that $f \in L^2[0, 2\pi]$.

END