UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in: MAT 3400 — Linear analysis with applications

Day of examination: Friday 6, December 2013

Examination hours: 09.00 – 13.00.

This problem set consists of 4 pages.

Appendices: None.

Permitted aids: None.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

There are 9 subproblems, with a total score of 100 points: you can score up to 11 points for each subproblem, except for the last subproblem (3c)) where the maximal score is 12 points.

Throughout the text, we let $\mathbb{K}$ denote either $\mathbb{R}$ or $\mathbb{C}$. All vector spaces are assumed to be vector spaces over $\mathbb{K}$.

Problem 1

Let X be a nonempty set and let $\mathcal{P}(X)$ denote the σ-algebra consisting of all subsets of X.

Let $\ell^\infty(X)$ denote the space of all bounded functions from X into $\mathbb{K}$, equipped with the uniform norm, that is, $\|f\|_u = \sup\big\{|f(x)| \mid x \in X\big\}$ when $f \in \ell^\infty(X)$.

a) Let μ denote a measure on $\mathcal{P}(X)$ satisfying $\mu(X) < \infty$. Explain why every $f \in \ell^\infty(X)$ is integrable w.r.t. μ and check that the linear map $I_\mu : \ell^\infty(X) \to \mathbb{K}$ defined by

$$I_\mu(f) = \int f \, d\mu \,, \quad f \in \ell^\infty(X),$$

is bounded, with $\|I_\mu\| = \mu(X)$.

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b) Assume a map $I : \ell^\infty(X) \to \mathbb{K}$ is linear, bounded, and satisfies that $I(f) \geq 0$ whenever $f \in \ell^\infty(X)$, $f \geq 0$.

Define $\nu : \mathcal{P}(X) \to [0, \infty)$ by

$$\nu(A) = I(\chi_A), \quad A \subset X,$$

where χ_A denotes the characteristic (or indicator) function of A in X.

Show that ν satisfies the following properties:

i) $\nu(\emptyset) = 0$.

ii) $\nu(\cup_{j=1}^n A_j) = \sum_{j=1}^n \nu(A_j)$ when $n \in \mathbb{N}$ and $A_1, \ldots, A_n \subset X$ are (pairwise) disjoint.

iii) $\nu(X) < \infty$.

Can you state an additional condition on I that will ensure that ν becomes a measure on $\mathcal{P}(X)$ such that $I = I_\nu$?

[*You don't have to give any argument, just write down your proposal for a suitable condition*].

c) Assume that $\nu : \mathcal{P}(X) \to [0, \infty)$ is a map satisfying the properties *i*), *ii*) and *iii*) mentioned in part *b*).

Show that there exists a linear, bounded map $I : \ell^\infty(X) \to \mathbb{K}$ such that $I(\chi_A) = \nu(A)$ for all $A \subset X$ and $\|I\| = \nu(X)$.

Hint: Let $\mathcal{E}$ denote the subspace of $\ell^\infty(X)$ consisting of all simple functions on X, that is, $\mathcal{E} = \text{Span}\,\{\chi_A \mid A \subset X\}$. Consider the map $I_0 : \mathcal{E} \to \mathbb{K}$ defined by

$$I_0(g) = \sum_{j=1}^{n} \lambda_j \, \nu(A_j)$$

when $g = \sum_{j=1}^n \lambda_j \, \chi_{A_j}$ denotes the standard representation of $g \in \mathcal{E}$. You may take as granted that I_0 is linear. Start by showing that I_0 is bounded.

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Problem 2

Let $(X, \mathcal{A}, \mu)$ be a measure space.

We set $\overline{\mathcal{M}}^+ = \{g : X \to [0, \infty] \mid g \text{ is } \mathcal{A}\text{-measurable}\}$ and let $\mathcal{L}^1$ denote the space of $\mathcal{A}$-measurable, $\mathbb{K}$-valued functions on X that are integrable with respect to μ.

Assume that f is an $\mathcal{A}$-measurable, $\mathbb{K}$-valued function on X such that there exists a sequence $\{h_j\}_{j\in\mathbb{N}}$ in $\mathcal{L}^1$ satisfying

i) $f(x) = \sum_{j=1}^{\infty} h_j(x)$ for μ-almost all x in X,

ii) $\sum_{j=1}^{\infty} \big(\int |h_j| \, d\mu\big) < \infty$.

Let $g \in \overline{\mathcal{M}}^+$ be given by $g = \sum_{j=1}^{\infty} |h_j|$.

a) Show that g is integrable w.r.t. μ.

b) Show that $f \in \mathcal{L}^1$ and $\int f \, d\mu = \sum_{j=1}^{\infty} \big(\int h_j \, d\mu\big)$.

c) We consider now the case where $X = [-1, 1]$, $\mathcal{A}$ denotes the Lebesgue-measurable subsets of $[-1, 1]$ and μ denotes the Lebesgue measure on $\mathcal{A}$.

As is well known from elementary calculus, the power series $\sum_{k=1}^{\infty} \frac{1}{k} x^k$ is convergent when $x \in [-1, 1)$, and divergent for $x = 1$.
We let $f : [-1, 1] \to \mathbb{R}$ denote the $\mathcal{A}$-measurable function defined by

$$f(x) = \sum_{k=1}^{\infty} \frac{1}{k} x^k \quad \text{when } x \in [-1, 1)\,, \text{ while } f(1) = 0\,.$$

Check that $f \in \mathcal{L}^1$. Then verify that

$$\int_{[-1,1]} f \, d\mu = \sum_{m=1}^{\infty} \frac{1}{m\,(2m+1)}\,.$$

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Problem 3

Let H be a Hilbert space, $H \neq \{0\}$, and $B(H)$ denote the space of all bounded linear operators from H into itself.

We consider $S, T \in B(H)$ and assume throughout this exercise that S and T commutes with each other, that is, we have $ST = TS$.

We recall that a subset M of H is said to be invariant under T if $T(u) \in M$ for all $u \in M$.

a) Let M be a closed subspace of H, $\lambda \in \mathbb{K}$ and set

$$E_\lambda^T = \{v \in V \mid T(v) = \lambda v\}\,.$$

Show that M is invariant under T if and only if $M^\perp$ is invariant under T^*. Show also that E_λ^T is invariant under S.

b) Assume that S is self-adjoint and T has an eigenvalue $\lambda \in \mathbb{K}$ such that the associated eigenspace E_λ^T is finite-dimensional.

Show that there exists an orthonormal basis for E_λ^T that consists of vectors that are also eigenvectors for S.

c) Assume that S is self-adjoint, T is compact and self-adjoint, and H is separable.

Show that there exists an orthonormal basis $\mathcal{B}$ for H consisting of eigenvectors for ST. Moreover, if we also assume that T is one-to one, explain why this orthonormal basis $\mathcal{B}$ may be chosen to consist of vectors that are eigenvectors for both S and T.

THE END