UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in MAT3400/4400 — Linear analysis with applications.

Day of examination: Friday, December 5, 2014.

Examination hours: 9.00 – 13.00.

This problem set consists of 3 pages.

Appendices: None.

Permitted aids: None.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Note: You must justify all your answers!

Problem 1

Let $(\Omega, \mathcal{A}, \mu)$ be a measure space. For A in $\mathcal{A}$ we let χ_A denote the characteristic function of A. For $1 \leq p < \infty$ let $\mathcal{L}^p(\Omega, \mathcal{A}, \mu)$ denote the space of complex-valued, $\mathcal{A}$-measurable functions g on Ω such that $\int_\Omega |g|^p \, d\mu < \infty$.

1a

(15 points) Let $\{f_n\}_{n \in \mathbb{N}}$ be a sequence of $\mathcal{A}$-measurable functions on Ω with values in $[0, \infty]$ for every $n \in \mathbb{N}$. Use the monotone convergence theorem to show that

$$\int_\Omega \Big(\sum_{n=1}^{\infty} f_n \Big) \, d\mu = \sum_{n=1}^{\infty} \Big(\int_\Omega f_n \, d\mu \Big).$$

1b

(10 points) Assume that there exists a sequence of pairwise disjoint sets $\{A_n\}_{n \in \mathbb{N}}$ in $\mathcal{A}$ such that $1 \leq \mu(A_n) < \infty$ for all $n \in \mathbb{N}$. Let

$$f = \sum_{n=1}^{\infty} \frac{1}{n} \mu(A_n)^{-1} \chi_{A_n}.$$

Show that $f \notin \mathcal{L}^1(\Omega, \mathcal{A}, \mu)$ and that $f \in \mathcal{L}^2(\Omega, \mathcal{A}, \mu)$. Give an example of a measure space $(\Omega, \mathcal{A}, \mu)$ and sets $\{A_n\}_{n \in \mathbb{N}}$ where the hypotheses are satisfied.

1c

(10 points) Assume that $\mu(\Omega) < \infty$. Use the fact that $1 \in \mathcal{L}^2(\Omega, \mathcal{A}, \mu)$ to show that $\mathcal{L}^2(\Omega, \mathcal{A}, \mu) \subseteq \mathcal{L}^1(\Omega, \mathcal{A}, \mu)$.

(Continued on page 2.)

Problem 2

In this problem the following are assumed known: if $(\Omega, \mathcal{A}, \mu)$ is a measure space, then $L^2(\mu)$ is the Hilbert space of complex-valued, $\mathcal{A}$-measurable, square-integrable functions on Ω (where functions that are equal μ-almost everywhere are identified) with inner product given by

$$(g_1, g_2) = \int_\Omega g_1(x)\overline{g_2(x)}\, d\mu(x)$$

for $g_1, g_2 \in L^2(\mu)$. A sequence $\{e_k\}_{k\in\mathbb{N}}$ is an orthonormal basis for $L^2(\mu)$ if and only if Parseval's identity $\|g\|_2^2 = \sum_{k\in\mathbb{N}} |(g, e_k)|^2$ is valid for each $g \in L^2(\mu)$.

2a

(10 points) Consider the measure space $[0, 1]$ with the Borel σ-algebra $\mathcal{B}_{[0,1]}$ and the Lebesgue measure λ restricted to $\mathcal{B}_{[0,1]}$. Suppose that $\{u_m\}_{m\in\mathbb{N}}$ and $\{v_n\}_{n\in\mathbb{N}}$ are two orthonormal bases in $L^2(\lambda)$. Show that the functions $\{w_{n,m}\}_{n,m\in\mathbb{N}}$ defined by $w_{n,m}(x, y) = u_m(x)v_n(y)$ for $(x, y) \in [0, 1] \times [0, 1]$ form an orthonormal family in $L^2(\lambda \times \lambda)$ where $\lambda \times \lambda$ is the product measure on $\mathcal{B}_{[0,1]} \otimes \mathcal{B}_{[0,1]}$.

2b

(10 points) Let $f \in L^2(\lambda\times\lambda)$. Show that for each $n \in \mathbb{N}$, the function defined by

$$F_n(x) = \int_{[0,1]} f(x, y)\overline{v_n(y)}\, d\lambda(y)$$

for λ-almost all $x \in [0, 1]$ is in $L^2(\lambda)$. Show that $\big(f, w_{n,m}\big)$, computed as an inner product in $L^2(\lambda \times \lambda)$, is equal to (F_n, u_m), computed as an inner product in $L^2(\lambda)$, for every $n, m \in \mathbb{N}$. (Hint: Use Fubini's theorem and the Cauchy-Schwarz inequality. To apply Fubini's theorem, refer to Problem 1c).

2c

(10 points) Conclude that $\{w_{n,m}\}_{n,m\in\mathbb{N}}$ forms an orthonormal basis for $L^2(\lambda \times \lambda)$. (Hint: Use Parseval's identity for each of the bases $\{u_m\}_m$ and $\{v_n\}_n$ and Problem 1a).

Problem 3

Let H be a Hilbert space over $\mathbb{C}$ with inner product $(\cdot, \cdot)$ and let $B(H)$ denote the space of bounded linear operators from H to H. Let $I \in B(H)$ denote the identity operator $I(x) = x$ for all $x \in H$.

(Continued on page 3.)

An operator $R \in B(H)$ is called an orthogonal reflection if there exists a closed subspace M of H such that

$$x + R(x) \in M \text{ and } x - R(x) \in M^{\perp} \text{ for every } x \in H.$$

3a

(15 points) Prove that if R is an orthogonal reflection, then $\frac{1}{2}(R + I)$ is an orthogonal projection.

3b

(10 points) Prove conversely that if $R \in B(H)$ is such that $\frac{1}{2}(R + I)$ is an orthogonal projection in $B(H)$, then R is an orthogonal reflection.

3c

(10 points) Prove that R is an orthogonal reflection if and only if $R = R^*$ and $R^2 = I$.

END