UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in:	MAT3400/4400 — Linear analysis with applications.
Day of examination:	Friday, December 2, 2016.
Examination hours:	14.30 – 18.30.
This problem set consists of 4 pages.	
Appendices:	None.
Permitted aids:	None.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

The points in parentheses indicate the maximum score for each problem or subproblem. The maximum score is granted for a correct and complete solution to the respective question. If you are unable to solve a subproblem, you may assume the result of that problem when solving later problems. E.g., if you cannot solve problem 2a, you may assume the result of 2a when solving 2b.

Note: You must justify all your answers!

Problem 1 (weight 20 points)

Let $(\Omega, \mathcal{A})$ be a measurable space and let μ_1, μ_2 be two measures on $(\Omega, \mathcal{A})$. Define a map $\nu \colon \mathcal{A} \to [0, \infty]$ by

$$\nu(A) = \mu_1(A) + \mu_2(A).$$

1a (weight 10 points)

Show that ν is a measure on $(\Omega, \mathcal{A})$.

1b (weight 10 points)

Show that

$$\int_\Omega f \, d\nu = \int_\Omega f \, d\mu_1 + \int_\Omega f \, d\mu_2,$$

for all $\mathcal{A}$-measurable functions $f \colon \Omega \to [0, \infty]$.
Hint: Use bootstrapping.

(Continued on page 2.)

Problem 2 (weight 30 points)

Let $\mathcal{M}$ denote the σ-algebra of Lebesgue measurable subsets of $(0,1]$ and let λ denote the Lebesgue measure on $((0,1],\mathcal{M})$. Recall that for $p \in [1,\infty)$ we define

$$\mathcal{L}^p = \left\{ f\colon (0,1] \to \mathbb{C} \;\middle|\; f \text{ is } \mathcal{M}\text{-measurable and } \int_{(0,1]} |f|^p \, d\lambda < \infty \right\},$$

and

$$\mathcal{L}^\infty = \left\{ f\colon (0,1] \to \mathbb{C} \;\middle|\; \begin{array}{c} f \text{ is } \mathcal{M}\text{-measurable,} \\ |f| \le K \ \lambda\text{-a.e. for some } K \in \mathbb{R} \end{array} \right\}.$$

For each $a \in (0,1)$, define a function $g_a\colon (0,1] \to \mathbb{R}$ by

$$g_a(x) = \frac{1}{x^a},$$

and define a function $h\colon (0,1] \to \mathbb{R}$ by

$$h(x) = \ln(x).$$

2a (weight 10 points)

Let $p \in [1,\infty)$. Show that $g_a \in \mathcal{L}^p$ if $a \in \left(0, \frac{1}{p}\right)$.

2b (weight 10 points)

Show that $h \in \mathcal{L}^p$ for all $p \in [1,\infty)$.
Hint: You may use, without proof, that for all $x \in (0,1]$ and all $a \in (0,1)$ we have

$$|\ln(x)| \le \frac{1}{a x^a}.$$

2c (weight 10 points)

Show that $h \notin \mathcal{L}^\infty$.

Problem 3 (weight 20 points)

For each $n \in \mathbb{N}$ define a function $f_n\colon [0,\infty) \to \mathbb{R}$ by

$$f_n(x) = e^{-x} \cos\left(\frac{x}{n}\right).$$

Let $\mathcal{M}$ denote the σ-algebra of Lebesgue measurable subsets of $[0,\infty)$ and let λ denote the Lebesgue measure on $([0,\infty),\mathcal{M})$.

3a (weight 10 points)

Show that f_n is λ-integrable for all $n \in \mathbb{N}$.

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3b (weight 10 points)

Show that

$$\lim_{n\to\infty} \int_{[0,\infty)} f_n \, d\lambda = 1.$$

Problem 4 (weight 20 points)

Let H be a separable complex Hilbert space and let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal basis of H.

Define

$$W = \left\{ \sum_{n=1}^{k} \alpha_n e_n \;\middle|\; k \in \mathbb{N}, \alpha_n \in \mathbb{C} \text{ for all } 1 \le n \le k \right\}.$$

That is, W is the linear span of the basis vectors $\{e_n\}_{n=1}^{\infty}$. Note that $W \neq H$, but that W is dense in H.

Denote by $\mathbb{P}$ the set of prime numbers. For each $p \in \mathbb{P}$ we define a map $T_p \colon W \to H$ by

$$T_p\left(\sum_{n=1}^{k} \alpha_n e_n\right) = \sum_{n=1}^{k} \alpha_n e_{p^n}.$$

4a (weight 10 points)

Let $p \in \mathbb{P}$ be given. Show that map T_p is linear and bounded. Conclude that T_p extends to a linear operator $S_p \in B(H)$ that satisfies

$$S_p e_n = e_{p^n}, \qquad \text{for all } n \in \mathbb{N}.$$

4b (weight 10 points)

Denote by $O \in B(H)$ the zero operator and by $I \in B(H)$ the identity operator, i.e., $Ox = 0$ and $Ix = x$ for all $x \in H$.

Show that if $p, q \in \mathbb{P}$ and $p \neq q$ then

(i) $S_p^* S_q = O$, and

(ii) $S_p^* S_p = I$,

where S_p^* denotes the adjoint of S_p.

Problem 5 (weight 10 points)

Let $(\Omega, \mathcal{A}, \mu)$ be a measure space with $\mu(\Omega) < \infty$. Let ν be a measure on $(\Omega, \mathcal{A})$ such that

$$\nu(A) \le \mu(A), \qquad \text{for all } A \in \mathcal{A}.$$

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Show that there exists a function $g \in \mathcal{L}^2(\Omega, \mathcal{A}, \mu)$ such that

$$\nu(A) = \int_A g \, d\mu, \qquad \text{for all } A \in \mathcal{A}.$$

Hint: Show that

$$[f] \mapsto \int_\Omega f \, d\nu,$$

defines a bounded linear functional on $L^2(\Omega, \mathcal{A}, \mu)$.