UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: MAT3400/4400 — Linear analysis with applications

Day of examination: Friday 8. December 2017

Examination hours: 14.30 – 18.30

This problem set consists of 3 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

The points in parentheses indicate the maximum score for each problem or subproblem. The maximum score is granted for a correct and complete solution to the respective question. If you are unable to solve a subproblem, you may assume the result of that problem when solving later problems. E.g., if you cannot solve problem 2a, you may assume the result of 2a when solving 2b.

Note: You must justify all your answers!

Problem 1 (weight 20%)

Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and let $f\colon \Omega \to \mathbb{C}$ be an $\mathcal{A}$-measurable function. Define for each $n \in \mathbb{N}$ a set

$$A_n = \{x \in \Omega \mid |f(x)| \geq n\}.$$

1a (weight 5%)

Show that

$$\lim_{n\to\infty} \chi_{A_n}(x) = 0$$

for all $x \in \Omega$.

1b (weight 5%)

Prove that

$$\int_\Omega |f|\chi_{A_n}\, d\mu \geq n\mu(A_n),$$

for all $n \in \mathbb{N}$.

(Continued on page 2.)

1c (weight 10%)

Prove that if f is μ-integrable then

$$\lim_{n\to\infty} n\mu(A_n) = 0.$$

Problem 2 (weight 20%)

For each $n \in \mathbb{N}$ define a function $f_n\colon [1,\infty) \to \mathbb{R}$ by

$$f_n(x) = \frac{(\sin(x))^n}{x^2}.$$

Let $\mathcal{M}$ denote the σ-algebra of Lebesgue measurable subsets of $[1,\infty)$ and let λ denote the Lebesgue measure on $([1,\infty),\mathcal{M})$.

Prove that f_n is λ-integrable for all $n \in \mathbb{N}$ and show that

$$\lim_{n\to\infty} \int_{[1,\infty)} f_n \, d\lambda = 0.$$

Problem 3 (weight 10%)

Let $\mathcal{M}$ denote the σ-algebra of Lebesgue measurable subsets of $[0,1]$ and let λ denote the Lebesgue measure on $([0,1]),\mathcal{M})$. Suppose $\psi\colon \mathbb{R} \to \mathbb{R}$ is a function such that for all λ-integrable functions $f\colon [0,1] \to \mathbb{R}$ the composition $\psi \circ f$ is λ-integrable and

$$\psi\left(\int_{[0,1]} f \, d\lambda\right) \le \int_{[0,1]} \psi \circ f \, d\lambda.$$

Prove that whenever $x_1, x_2 \in \mathbb{R}$ and $a_1, a_2 \in [0,1]$ are such that $a_1 + a_2 = 1$ then

$$\psi(a_1 x_1 + a_2 x_2) \le a_1 \psi(x_1) + a_2 \psi(x_2).$$

Problem 4 (weight 30%)

Let $(\Omega, \mathcal{A}, \mu)$ be a measure space with $\mu(\Omega) < \infty$. Recall that we define

$$L^2(\Omega) = \left\{ f\colon \Omega \to \mathbb{C} \;\middle|\; f \text{ is } \mathcal{A}\text{-measurable and } \int_\Omega |f|^2 \, d\mu < \infty \right\},$$

and that in $L^2(\Omega)$ we identify functions that are equal μ-almost everywhere. Recall further that $L^2(\Omega)$ is a Hilbert space with inner product and norm given by

$$\langle f, g \rangle = \int_\Omega f\overline{g} \, d\mu, \qquad \text{and} \qquad \|f\|_2 = \left(\int_\Omega |f|^2 \, d\mu\right)^{\frac{1}{2}},$$

where $f, g \in L^2(\Omega)$.

For each $E \in \mathcal{A}$ we define an operator $P_E \in B(L^2(\Omega))$ given by

$$P_E f = f\chi_E,$$

for $f \in L^2(\Omega)$. You do not need to show that P_E is a bounded linear operator.

(Continued on page 3.)

4a (weight 10%)

Show that P_E is an orthogonal projection for all $E \in \mathcal{A}$.

4b (weight 10%)

Suppose $\{E_n\}_n \subseteq \mathcal{A}$ is a sequence of sets such that

$$E_1 \subseteq E_2 \subseteq E_3 \subseteq \cdots, \quad \text{and} \quad \bigcup_{n=1}^{\infty} E_n = \Omega.$$

Show that for all $f \in L^2(\Omega)$ we have

$$\lim_{n\to\infty} \|f - P_{E_n} f\|_2 = 0.$$

4c (weight 10%)

Let $I \in B(L^2(\Omega))$ be the identity operator, that is $If = f$ for all $f \in L^2$. Suppose, as in **4b**, that $\{E_n\}_n \subseteq \mathcal{A}$ is a sequence of sets such that

$$E_1 \subseteq E_2 \subseteq E_3 \subseteq \cdots, \quad \text{and} \quad \bigcup_{n=1}^{\infty} E_n = \Omega.$$

Suppose further that $\mu(\Omega \setminus E_n) > 0$ for all $n \in \mathbb{N}$. Show that the sequence $\{P_{E_n}\}_n$ does not converge to I in operator norm, that is

$$\|I - P_{E_n}\| \not\to 0 \text{ as } n \to \infty.$$

Problem 5 (weight 20%)

Let H be an infinite dimensional Hilbert space, and let $S, T \in B(H)$ be self-adjoint operators.

5a (weight 10%)

Show that if T is compact and $\|Sx\| \leq \|Tx\|$ for all $x \in H$, then S is compact.

5b (weight 10%)

Let $I \in B(H)$ denote the identity operator, that is $Ix = x$ for all $x \in H$. Show that if $I \leq S$, then S is not compact.

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