UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in: MAT3400/4400 — Linear analysis with applications

Day of examination: Thursday, June 14, 2018

Examination hours: 14.30 – 18.30

This problem set consists of 3 pages.

Appendices: None.

Permitted aids: None.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

The points in parentheses indicate the maximum score for each problem or subproblem. The maximum score is granted for a correct and complete solution to the respective question. If you are unable to solve a subproblem, you may assume the result of that problem when solving later problems. E.g., if you cannot solve problem 2a, you may assume the result of 2a when solving 2b.

Note: You must justify all your answers!

Problem 1 (weight 15 points)

Let $(\Omega, \mathcal{A}, \mu)$ be a measure space. For each integer $k \in \mathbb{Z}$, let $A_k \in \mathcal{A}$ and assume that $A_k \cap A_l = \varnothing$ whenever $k \neq l$. Set $A := \bigcup_{k \in \mathbb{Z}} A_k \in \mathcal{A}$.

Assume $f : \Omega \to \mathbb{C}$ is $\mathcal{A}$-measurable and integrable over A with respect to μ.

1a (weight 5 points)

Explain why f is integrable over A_k with respect to μ for each $k \in \mathbb{N}$.

1b (weight 10 points)

Show that

$$\int_A f \, d\mu = \lim_{n \to \infty} \sum_{k=-n}^{n} \int_{A_k} f \, d\mu \, .$$

(Continued on page 2.)

Problem 2 (weight 40 points)

Let $\mathcal{M}$ denote the σ-algebra of all Lebesgue measurable subsets of $\mathbb{R}$ and let λ denote the Lebesgue measure on $(\mathbb{R}, \mathcal{M})$. For any $E \subset \mathbb{R}$ and $a \in \mathbb{R}$, set

$$E - a := \big\{x - a \mid x \in E\big\}.$$

We recall that for all $E \in \mathcal{M}$ and $a \in \mathbb{R}$, we have $E - a \in \mathcal{M}$ and $\lambda(E - a) = \lambda(E)$.

2a (weight 15 points)

Let $a \in \mathbb{R}$ and $E \in \mathcal{M}$. Show that if $f : \mathbb{R} \to [0, \infty)$ is $\mathcal{M}$-measurable, then

$$\int_E f(x)\, d\lambda(x) = \int_{E-a} f(x+a)\, d\lambda(x).$$

Hint: Check this first when $f = \chi_A$ for some $A \in \mathcal{M}$.

Next, we recall that a function $g : \mathbb{R} \to \mathbb{C}$ is called *periodic* if there exists some $a > 0$ such that $g(x + a) = g(x)$ for all $x \in \mathbb{R}$.

2b (weight 10 points)

Assume $g : \mathbb{R} \to \mathbb{C}$ is $\mathcal{M}$-measurable and periodic.

Prove that g is Lebesgue integrable if and only if $g = 0$ λ-a-e.

2c (weight 15 points)

Let $h : \mathbb{R} \to \mathbb{R}$ be given by

$$h(x) = \frac{2 + \sin(x)}{\sqrt{x^2 + 1}} \quad \text{for all } x \in \mathbb{R}.$$

Determine for which $p \in [1, \infty)$ we have that $h \in \mathcal{L}^p(\mathbb{R}, \mathcal{M}, \lambda)$.

Problem 3 (weight 10 points)

Let $\mathbb{F}$ denote $\mathbb{R}$ or $\mathbb{C}$, and let X be a normed space over $\mathbb{F}$. Let $\varphi \in \mathcal{B}(X, \mathbb{F})$ and assume that $\varphi \neq 0$.

Show that there exists some $y \in X \setminus \{0\}$ such that

$$X = \ker(\varphi) \oplus N,$$

where $N := \text{Span}\{y\}$.

(Continued on page 3.)

Problem 4 (weight 25 points)

Let H be an infinite-dimensional Hilbert space (over $\mathbb{R}$ or $\mathbb{C}$) having a countable orthonormal basis $\mathcal{B} = \{u_n \mid n \in \mathbb{N}\}$.

4a (weight 5 points)

Let $T \in \mathcal{B}(H)$. Justify that for every $k \in \mathbb{N}$ we have

$$\big\langle T(u_k), T(u_k) \big\rangle = \sum_{n=1}^{\infty} \big| \big\langle T(u_k), u_n \big\rangle \big|^2 .$$

4b (weight 10 points)

Let $P \in \mathcal{B}(H)$ be a self-adjoint projection (i.e., we have $P^* = P = P^2$).

Assume that $\big\langle P(u_k), u_k \big\rangle \in \{0, 1\}$ for some $k \in \mathbb{N}$.

Show that $\big\langle P(u_k), u_n \big\rangle = 0$ for all $n \in \mathbb{N}$ such that $n \neq k$.

4c (weight 10 points)

Let $P \in \mathcal{B}(H)$ be a self-adjoint projection and let $D_P \in \mathcal{B}(H)$ denote the diagonal operator w.r.t. $\mathcal{B}$ which satisfies that

$$D_P(u_n) = \big\langle P(u_n), u_n \big\rangle \, u_n \quad \text{for all } n \in \mathbb{N}.$$

Show that if D_P is a projection, then we have $P = D_P$.

Problem 5 (weight 10 points)

Let H be an infinite-dimensional Hilbert space (over $\mathbb{R}$ or $\mathbb{C}$) and let $T \in \mathcal{B}(H)$ be compact. Explain what it means that T is compact. Then show that $\big\langle T(u_n), u_n \big\rangle \to 0$ as $n \to \infty$ whenever $\{u_n\}_{n \in \mathbb{N}}$ is an orthonormal sequence in H.

Good luck!