UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: MAT3400/4400 — Linear Analysis with Applications

Day of examination: Tuesday, 31 May 2022

Examination hours: 15.00 – 19.00

This problem set consists of 2 pages.

Appendices: None.

Permitted aids: All.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

All subproblems count equally. If there is a subproblem you cannot solve, you may still use the result in the sequel. All answers have to be substantiated.

Problem 1 (weight 10%)

Let $(X, \mathcal{A}, \mu)$ be a complete measure space. Suppose that f and g are functions from X to $\overline{\mathbb{R}}$ and set

$$N = \{x \in X \,:\, f(x) \neq g(x)\}.$$

Prove that if f is measurable and N is a null set, then g is also measurable.

Problem 2 (weight 20%)

(a) Suppose that $\{a_{j,k}\}_{k\geq 1}$ is an increasing sequence of nonnegative extended real numbers for each integer $j \geq 1$. Explain why

$$\lim_{k\to\infty} \sum_{j=1}^{\infty} a_{j,k} = \sum_{j=1}^{\infty} \lim_{k\to\infty} a_{j,k}.$$

Hint. Counting measure!

(b) Let $(X, \mathcal{A})$ be a measurable space and let $\{\mu_k\}_{k\geq 1}$ be a sequence of measures on $\mathcal{A}$ which enjoy the property that

$$\mu_1(A) \leq \mu_2(A) \leq \mu_3(A) \leq \cdots$$

for every A in $\mathcal{A}$. Show that the limit

$$\mu(A) = \lim_{k\to\infty} \mu_k(A)$$

defines a measure on $\mathcal{A}$.

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Problem 3 (weight 20%)

Let μ denote the Lebesgue measure on $\mathbb{R}$. For subsets A and B of $\mathbb{R}$, consider

$$A + B = \{a + b : a \in A \text{ and } b \in B\}.$$

(a) Suppose that A and B are subsets of $[0,1]$ and that $A + B$ is Lebesgue measurable. Prove that $\mu(A + B) \leq 2$.

(b) Find a subset C of $[0,1]$ such that $\mu(C) = 0$ and $\mu(C + C) = 2$.

Problem 4 (weight 10%)

Let H be a Hilbert space and let T be a linear operator on H with $\|T\| = 1$. Prove that if $T(x) = x$ for some vector x in H, then $T^*(x) = x$ as well.

Problem 5 (weight 30%)

Consider the Hilbert space $H = L^2([0,1], \mu)$, where μ is the Lebesgue measure.

(a) For $0 \leq a \leq 1$, consider the bounded linear functional on H defined by

$$\varphi_a(f) = \int_{[0,a)} f \, d\mu.$$

Find an element g_a in H such that $\varphi_a(f) = \langle f, g_a \rangle$ and compute $\|\varphi_a\|$.

(b) Let n be a fixed positive integer. Set $\mathcal{U}_n = \{u_{n,k}\}_{k=1}^{n}$, where

$$u_{n,k}(x) = \begin{cases} 1, & \text{if } \frac{k-1}{n} \leq x < \frac{k}{n}, \\ 0, & \text{else.} \end{cases}$$

Consider the bounded linear operator $T_n \colon H \to H$ defined by

$$T_n f(x) = \varphi_{\frac{k}{n}}(f) \quad \text{for} \quad \frac{k-1}{n} \leq x < \frac{k}{n} \quad \text{and} \quad T_n f(1) = 0.$$

Prove that $T_n(H) = \operatorname{Span}(\mathcal{U}_n)$. What is $\operatorname{rank}(T_n)$?

(c) Let T be the linear operator on H defined by

$$Tf(x) = \int_{[0,x)} f \, d\mu.$$

Prove that T is compact.

Problem 6 (weight 10%)

Let T be a compact self-adjoint operator on a Hilbert space H and assume that $\ker T = 0$. Let $\{\lambda_j\}_{j \geq 1}$ denote the sequence of eigenvalues of T repeated according to their multiplicity. Define

$$a = \inf_{j \geq 1} \lambda_j \qquad \text{and} \qquad b = \sup_{j \geq 1} \lambda_j.$$

Let W_T denote the numerical range of T. Prove that $W_T \subseteq [a, b]$.

THE END