$\mathcal{L}^p$ vs L^p

Throughout this note we fix a measure space $(\Omega, \mathcal{A}, \mu)$. We first recall some definitions and results from [MW13].

Definition 1. Let $p \in (0, \infty)$. The set of *p-integrable functions* is defined to be

$$\mathcal{L}^p(\Omega, \mathcal{A}, \mu) = \left\{ f\colon \Omega \to \mathbb{C} \;\middle|\; f \text{ is } \mathcal{A}\text{-measurable and } \int_\Omega |f|^p \, d\mu < \infty \right\}.$$

For an $\mathcal{A}$-measurable function f we define the p-norm of f as

$$\|f\|_p = \left(\int_\Omega |f|^p \, d\mu \right)^{\frac{1}{p}}.$$

Note that the p-integrable functions are exactly those functions for which $\|f\|_p < \infty$.

Definition 2. For an $\mathcal{A}$-measurable function f we define the ∞-norm of f (also called the essential supramum of f) by

$$\|f\|_\infty = \inf \{ M \in \mathbb{R}^* \mid |f| \leq M \ \mu\text{-a.e.} \}.$$

The set of *essentially bounded functions* is defined to be

$$\mathcal{L}^\infty(\Omega, \mathcal{A}, \mu) = \{ f\colon \Omega \to \mathbb{C} \mid f \text{ is } \mathcal{A}\text{-measurable and } \|f\|_\infty < \infty \}.$$

Proposition 3. *Let $p \in [1, \infty]$. Then $\mathcal{L}^p(\Omega, \mathcal{A}, \mu)$ is a linear space and the p-norm defines a seminorm on $\mathcal{L}^p(\Omega, \mathcal{A}, \mu)$.*

Proof. By [MW13, Proposition 13.4b] $\mathcal{L}^p(\Omega, \mathcal{A}, \mu)$ is a linear space and [MW13, Proposition 13.4, Theorem 13.10] show that $\| \cdot \|_p$ is a seminorm on $\mathcal{L}^p(\Omega, \mathcal{A}, \mu)$. □

Lemma 4. *Let $p \in [1, \infty]$ and let $f\colon \Omega \to \mathbb{C}$ be $\mathcal{A}$-measurable. Then $\|f\|_p = 0$ if and only if $f = 0$ μ-a.e..*

Proof. Suppose that $p < \infty$. The case of $p = \infty$ is left as an exercise for the reader. By [MW13, Exercise 5.52] we get that

$$\|f\|_p = 0 \iff \int_\Omega |f|^p \, d\mu = 0 \iff |f|^p = 0 \ \mu\text{-a.e.} \iff f = 0 \ \mu\text{-a.e.}.$$ □

Whenever $(\Omega, \mathcal{A}, \mu)$ is such that there are functions that are 0 almost everywhere without being 0 everywhere, the above shows that $\| \cdot \|_p$ is not a norm. To remedy this, we will identify functions that are equal almost everywhere. Formally this is done by introduce an equivalence relation on $\mathcal{L}^p(\Omega, \mathcal{A}, \mu)$ and then taking a quotient space.

Definition 5. Let $p \in [1, \infty]$. We say that two functions $f, g \in \mathcal{L}^p(\Omega, \mathcal{A}, \mu)$ are equivalent if $f = g$ μ-a.e.. We write

$$f \sim g \iff f = g \ \mu\text{-a.e.}.$$

Note that by Lemma 4 $f \sim g$ if and only if $\|f - g\|_p = 0$.

Lemma 6. *Let $p \in [1, \infty]$. The relation defined in Definition 5 is an equivalence relation.*

Proof. We have to check that $\sim$ is reflexive, symmetric, and transitive. That it is reflexive and symmetric is obvious. To show transitivity let $f, g, h \in \mathcal{L}^p(\Omega, \mathcal{A}, \mu)$ be such that $f \sim g$ and $g \sim h$. Then,

$$\|f - h\|_p = \|f - g + g - h\|_p \leq \|f - g\|_p + \|g - h\|_p = 0$$

So $f \sim h$. □

We will write $[f]$ for the equivalence class of a function $f \in \mathcal{L}^p(\Omega, \mathcal{A}, \mu)$. That is

$$[f] = \{g \in \mathcal{L}^p(\Omega, \mathcal{A}, \mu) \mid f \sim g\}.$$

Definition 7. Let $p \in [1, \infty]$ and let $\sim$ be the equivalence relation from Definition 5. Define $L^p(\Omega, \mathcal{A}, \mu)$ to be the quotient space $L^p(\Omega, \mathcal{A}, \mu)/\sim$. That is

$$L^p(\Omega, \mathcal{A}, \mu) = \{[f] \mid f \in \mathcal{L}^p(\Omega, \mathcal{A}, \mu)\}.$$

Lemma 8. *Let $p \in [1, \infty]$. Under the operations $[f] + [g] = [f + g]$ and $\alpha[f] = [\alpha f]$, $f, g \in \mathcal{L}^p(\Omega, \mathcal{A}, \mu)$, $\alpha \in \mathbb{C}$, $L^p(\Omega, \mathcal{A}, \mu)$ is a linear space.*

Proof. First we must verify that the operations are indeed well defined, that is that if $f, g, f', g' \in \mathcal{L}^p(\Omega, \mathcal{A}, \mu)$ and $f \sim f'$, $g \sim g'$ then $f + g \sim f' + g'$. We see that

$$\|(f + g) - (f' - g')\|_p = \|(f - f') + (g - g')\|_p \leq \|f - f'\|_p + \|g - g'\|_p = 0 + 0.$$

So the addition is well defined. A similar but simpler computation shows that scalar multiplication is well defined.

It is tedious, but not hard, to check that operations satisfy all the vector space axioms. □

Lemma 9. *Let $p \in [1, \infty)$. The p-norm given by*

$$\|[f]\|_p = \|f\|_p = \left(\int_\Omega |f|^p \, d\mu\right)^{\frac{1}{p}},$$

defines a norm on $L^p(\Omega, \mathcal{A}, \mu)$.

Proof. Again we first verify that the norm is well defined. Suppose $f \sim g$. Since $f = g$ μ-a.e., we have

$$\|f\|_p = \left(\int_\Omega |f|^p \, d\mu \right)^{\frac{1}{p}} = \left(\int_\Omega |g|^p \, d\mu \right)^{\frac{1}{p}} = \|g\|_p.$$

So the p-norm on $L^p(\Omega, \mathcal{A}, \mu)$ is well defined.

Using that the p-norm on $\mathcal{L}^p(\Omega, \mathcal{A}, \mu)$ is a seminorm, we get that the p-norm on $L^p(\Omega, \mathcal{A}, \mu)$ is a seminorm. It then follows from Lemma 4 that it is in fact a norm. □

When working with L^p we will often abuse notation and simply talk about functions in L^p rather than equivalence classes of functions. As long as our discussion is limited to integrals this should not cause too much confusion. But be aware that this can be tricky, for instance it does not make sense to say that a function in L^p is continuous, since $f \sim g$ and f continuous does not imply that g is continuous (can you give an example?).

References

[MW13] John N. McDonald and Neil A. Weiss. *A course in real analysis*. Academic Press, Inc., San Diego, CA, second edition, 2013. Biographies by Carol A. Weiss.