Exercise 13.26

Let K be a closed lienar subspace of a Hilbert space H and P_K the associated orthogonal projection. Verify the following properties

(a) P_K *is linear.*

(b) $\|P_K(x)\| \leq \|x\|$, *for all* $x \in H$.

(c) $P_K \circ P_K = P_K$,

(d) $P_K^{-1}(\{0\}) = K^\perp$. $(P_K^{-1}(\{0\}) = \ker P_K)$.

(e) The range of P_K *is* K.

(f) $P_{K^\perp} = I - P_K$, *where* I *is the identity operator on* H.

(g) Each $x \in H$ *can be written uniquely as* $x = y + y'$, *where* $y \in K$ *and* $y' \in K^\perp$.

Proof.

Throughout we shall make repeated use of Theorem 13.2 without explicitly mentioning it. In particular we will use that in order to show that $P_K x = z$ for $x, z \in H$, it suffices to show

$$z \in K \quad \text{and} \quad \langle x - z, y \rangle = 0 \text{ for all } y \in K.$$

Part (a): Suppose that $x, x' \in H$. We wish to show that $P_K(x + x') = P_K x + P_K x'$. First we note that since K is a subspace of H we have that $P_K x + P_K x'$ belongs to K. For any $y \in K$ we see that

$$\begin{aligned}\langle (x + x') - (P_K x + P_K x'), y \rangle &= \langle (x - P_K x) + (x' - P_K x'), y \rangle \\ &= \langle (x - P_K x), y \rangle + \langle (x' - P_K x'), y \rangle = 0 + 0.\end{aligned}$$

Therefore $P_K(x + x') = P_K x + P_K x'$.

Suppose now $\alpha \in \mathbb{C}$, we then wish to show that $P_K(\alpha x) = \alpha P_K x$. Again $\alpha P_K x$ is in K, and for any $y \in K$ we have

$$\langle \alpha x - \alpha P_K x, y \rangle = \alpha \langle x - P_K x, y \rangle = \alpha 0 = 0.$$

So $\alpha P_K x = P_K(\alpha x)$.

Part (b): Suppose that $x \in H$. Using (in the last step) that $P_K x \in K$ we have that

$$\begin{aligned}\|P_K x\|^2 &= \langle P_K x, P_K x \rangle = \langle (P_K x - x) + x, P_K x \rangle \\ &= \langle x, P_K x \rangle - \langle x - P_K x, P_K x \rangle = \langle x, P_K x \rangle.\end{aligned}$$

In particular $\langle x, P_K x\rangle$ is a non-negative real number. Thus by the Cauchy inequality

$$\|P_K x\|^2 = \langle x, P_K x\rangle = |\langle x, P_K x\rangle| \leq \|x\|\|P_K x\|.$$

If $P_K x \neq 0$, then deviding both sides by $\|P_K x\|$ gives

$$\|P_K x\| \leq \|x\|,$$

as desired. If $P_K x = 0$ then the inequality holds for trivial reasons.

Part (c) and (e): Both follow once we notice that for $y \in K$ we have $P_K y = y$, since y clearly is the point in K nearest to y.

Part (d): Suppose first that $z \in \ker P_K$. Then for any $y \in K$ we have

$$0 = \langle z - P_K z, y\rangle = \langle z, y\rangle.$$

Hence z is in $P_K^{\perp}$. Suppose instead that $z \in P_K^{\perp}$. Then for any $y \in K$ we have

$$0 = \langle z, y\rangle = \langle z - 0, y\rangle.$$

Since 0 is in K it follows that $P_K z = 0$.

Part (f): Let $x \in H$ be given. We claim that $P_{K^\perp} x = x - P_K x$. For every $y \in K$ we have

$$\langle x - P_K x, y\rangle = 0,$$

hence $x - P_K x \in K^{\perp}$. For every $z \in K^{\perp}$ we have

$$\langle x - (x - P_K x), z\rangle = \langle P_K x, z\rangle = 0.$$

Hence $x - P_K x$ is the unique point in $K^{\perp}$ nearest to x, so $P_{K^\perp} x = x - P_K x$.

Part (g): For any $x \in H$ we have, by (f), that

$$x = P_K x + P_{K^\perp} x.$$

So each x has at least one decomposition $x = y + y'$ with $y \in K, y' \in K^{\perp}$. To see that this decomposition is unique suppose that we also have $x = z + z'$ with $z \in K, z' \in K^{\perp}$. Then

$$y + y' = x = z + z'.$$

So

$$y - z = z' - y'.$$

Since K is a subspace $y - z \in K$, and similarly $z' - y' \in K^{\perp}$. Hence $y - z \in K \cap K^{\perp}$ and $z' - y' \in K \cap K^{\perp}$. As $K \cap K^{\perp} = \{0\}$ (why?), this shows that $y = z$ and $z' = y'$, i.e., the decomposition is unique.