Problem 1

Define $f\colon \mathbb{R}^2 \to \mathbb{R}$ by $f(x, y) = x \cdot y^2$. Let $A = [-1, 1] \times [-1, 1]$. Compute

$$\int_A f(x, y)\, d(\lambda \otimes \lambda)(x, y).$$

Problem 2

Let $A = \left\{(x, y) \in \mathbb{R}^2 \;\middle|\; 0 < y < x < 1\right\}$ (draw A as a subset of $\mathbb{R}^2$). Compute

$$\int_A (x - y)^\alpha\, d(\lambda \otimes \lambda)(x, y),$$

for $\alpha \in \mathbb{R}$.

Problem 3

Let $f\colon \mathbb{R} \to [0, \infty]$ be a measurable function. Define

$$G_f = \left\{(x, y) \in \mathbb{R}^2 \;\middle|\; 0 \le y \le f(x)\right\}.$$

Show that G_f is $\mathcal{M} \otimes \mathcal{M}$-measurable and that

$$\int_{\mathbb{R}} f\, d\lambda = (\lambda \otimes \lambda)(G_f).$$

Note: This shows that the integral of f is exactly the area under the graph.

Problem 4

Consider the measure spaces $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$, where $\mathbb{N}$ are the positive integers and μ is the counting measure. Define a function $f\colon \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ by

$$f(x, y) = \begin{cases} 1 & \text{if } x = y, \\ -1 & \text{if } x = y + 1, \\ 0 & \text{otherwise} \end{cases}.$$

Show that $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$ is σ-finite, and that both the iterated integrals

$$\int_{\mathbb{N}} \left(\int_{\mathbb{N}} f(x, y)\, d\mu(y) \right) d\mu(x) \quad \text{and} \quad \int_{\mathbb{N}} \left(\int_{\mathbb{N}} f(x, y)\, d\mu(x) \right) d\mu(y),$$

exist but that they have different values. Why doesn't this contradict Fubini's Theorem?