UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in	MAT3400/4400 — Linear analysis with applications.
Day of examination:	Monday, December 7, 2015.
Examination hours:	9.00 – 13.00.
This problem set consists of 3 pages.	
Appendices:	None.
Permitted aids:	None.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Note: You must justify all your answers!

Problem 1

(15 points) Let $X = [0, \infty)$ with the Borel σ-algebra $\mathcal{B}_X$ and the Lebesgue measure λ on X. Compute the limit

$$\lim_{n\to\infty} \int_{X\times X} e^{-(x+y+\frac{x^2y^2}{n})}\, d(\lambda\otimes\lambda)(x,y).$$

Problem 2

2a

(15 points) Let $(\Omega, \mathcal{A}, \mu)$ be a measure space. For $1 \le p < \infty$ let $\mathcal{L}^p(\mu)$ be the vector space of functions $h : \Omega \to \mathbb{C}$ which are $\mathcal{A}$-measurable and satisfy $\int_\Omega |h|^p \, d\mu < \infty$. Suppose that $\{f_n\}_{n\in\mathbb{N}}$ is a sequence in $\mathcal{L}^p(\mu)$ and f is an $\mathcal{A}$-measurable function such that f_n converges to f μ-a.e. If there is a nonnegative function $g \in \mathcal{L}^p(\mu)$ such that $|f_n| \le g$ μ-a.e. for all n, show that f_n converges in $\mathcal{L}^p(\mu)$ to f, that is

$$\lim_{n\to\infty} \int_\Omega |f_n - f|^p \, d\mu = 0.$$

(Hint: Note that $Cg^p \in \mathcal{L}^1(\mu)$ for any constant $C > 0$.)

2b

(10 points) Consider now the measure space $\Omega = [1, \infty)$ with the σ-algebra $\mathcal{B}_\Omega$ of Borel subsets of $[1, \infty)$ and the Lebesgue measure λ on $\mathcal{B}_\Omega$. Let

(Continued on page 2.)

$v_a : [1,\infty) \to \mathbb{R}$ be the function $v_a(x) = \frac{1}{x^a}$ for a a real number. It is assumed known that $v_a \in \mathcal{L}^1(\lambda)$ if and only if $a > 1$. For each $n \geq 1$ define

$$f_n(x) = \frac{n}{n\sqrt{x}+1} \text{ for } x \in [1,\infty).$$

Prove that $f_n \in \mathcal{L}^p(\lambda)$ when $2 < p < \infty$.

2c

(10 points) With assumptions as in problem 2b, let $2 < p < \infty$. Prove that f_n converges in $L^p(\lambda)$ and find the limit.

Problem 3

(10 points) Consider the measure space $[0,1]$ with the Borel σ-algebra $\mathcal{B}$ on $[0,1]$ and the Lebesgue measure λ. Let $L^2(\lambda)$ be the Hilbert space of $\mathbb{C}$-valued, $\mathcal{B}$-measurable, square-integrable functions on $[0,1]$ (where functions that are equal λ-a.e. are identified). Suppose that $u : [0,1] \to \mathbb{C}$ is a $\mathcal{B}$-measurable function such that $\sup\{|u(x)| \mid x \in [0,1]\} < \infty$. Let U be the bounded linear operator on $L^2(\lambda)$ given by $U(f)(x) = u(x)f(x)$ for $x \in [0,1]$.

Prove that U is an isometry in $B(L^2(\lambda))$ if and only if $|u(x)| = 1$ λ-a.e. on $[0,1]$. Show in this case that U is a unitary operator.

Problem 4

Let H be an infinite dimensional Hilbert space over $\mathbb{C}$. For $B \in B(H)$ the equalities of closed subspaces of H are given: $\ker(B^*B) = \ker(B)$ and $\overline{\operatorname{Im}(B^*B)} = \overline{\operatorname{Im}(B)}$.

4a

(5 points) Prove that if $S \in B(H)$ is a positive operator, meaning that S is self-adjoint and $(S(x) \mid x) \geq 0$ for all $x \in H$, then any eigenvalue of S is a non-negative real number.

4b

(10 points) Let B be a compact operator in $B(H)$. Suppose that $\overline{\operatorname{Im}(B^*B)}$ is infinite dimensional. Justify the claim: the non-zero eigenvalues of the operator B^*B form a sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ of positive real numbers converging to 0 and $\overline{\operatorname{Im}(B^*B)}$ has an orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$ consisting of eigenvectors corresponding to $\{\lambda_n\}$.

(Continued on page 3.)

4c

(10 points) With the assumptions from problem 4b, let $f_n = \frac{1}{\sqrt{\lambda_n}} B(e_n)$ for $n \in \mathbb{N}$. Prove that $\{f_n\}_{n\in\mathbb{N}}$ is an orthonormal sequence in H such that for each $x \in H$,

$$B(x) = \sum_{n=1}^{\infty} \sqrt{\lambda_n}(x \mid e_n) f_n.$$

4d

(15 points) Let H be $l^2(\mathbb{N})$ and let i denote the complex number with $i^2 = -1$. Show that $B(\{a_n\}_n) = \{\frac{i^n}{n} a_n\}_n$ defines a compact operator in $B(l^2(\mathbb{N}))$. Find sequences $\{\lambda_n\}_{n\in\mathbb{N}}$, $\{e_n\}_{n\in\mathbb{N}}$ and $\{f_n\}_{n\in\mathbb{N}}$ as in problem 4c.

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