

Measures

Fingertip knowledge: What is a measure?

Five minute problem: Fix a point $y \in \mathbb{R}$. Show that the function $\delta_y: \mathcal{P}(\mathbb{R}) \rightarrow [0, \infty]$ given by

$$\delta_y(A) = \begin{cases} 1 & \text{if } y \in A \\ 0 & \text{if } y \notin A \end{cases},$$

is a measure on $(\mathbb{R}, \mathcal{P}(\mathbb{R}))$. This measure is called the *Dirac measure* at the point y .

Measurable functions

Fingertip knowledge: If (Ω, \mathcal{A}) is a measurable space, then what does it mean for a function f from Ω to \mathbb{C} or \mathbb{R}^* to be measurable?

Five minute problem: Show that the function $f: \mathbb{R} \rightarrow \mathbb{R}$ given by

$$f(x) = \begin{cases} \sin(x) & \text{if } x > 0 \\ \chi_{\mathbb{Q}}(x) & \text{if } x \leq 0 \end{cases},$$

is measurable.

Integration of non-negative functions

Fingertip knowledge: How is the integral a non-negative function defined?

Five minute problem: Fix a point $y \in \mathbb{R}$ and consider the Dirac measure δ_y . Show that

$$\int f d\delta_y = f(y),$$

for all functions $f: \mathbb{R} \rightarrow [0, \infty)$.

Lebesgue integration and Riemann integration

Fingertip knowledge: When do the Riemann and Lebesgue integrals agree?

Five minute problem: Compute

$$\int_{[0, \pi]} \sin(x) d\lambda(x).$$

Integration of complex valued functions

Fingertip knowledge: When is a complex valued function integrable? How is the integral a real or complex values function defined?

Five minute problem: Consider the function $f: (1, \infty) \rightarrow \mathbb{R}$ given by

$$f(x) = \frac{\sin(x)}{x^2}.$$

Is f integrabel?

Interchanging limits and integrals

Fingertip knowledge: Under what conditions can we interchange a limit and an integration sign? (monotone convergence and dominated convergence)

Five minute problem: Let $f_n(x) = \sin^n(x)$. Show

$$\lim_n \int_{[0,2\pi]} f_n d\lambda = 0.$$