Exercise 1 — Limits

For each $n \in \mathbb{N}$ let $f_n(x) = \sin^n(x)$ and let

$$a_n = \int_0^\pi f_n(x)\, dx.$$

(a) Show that (a_n) is a decreasing sequence.

(b) Show that $\lim_n a_n = 0$.

Exercise 2 — The distance between curves

Consider the set $C([0,\pi])$ of continuous function from $[0,\pi]$ to $\mathbb{R}$.

(a) Show that for any $h \in C([0,\pi]])$ that only takes strictly positive values, we can define a norm on $C([0,\pi])$ by

$$\|f\|_h = \int_0^\pi |f(x)| h(x)\, dx.$$

Abusing notation slightly we will let 1 denote the function that constantly takes the value 1, so that

$$\|f\|_1 = \int_0^\pi |f(x)|\, dx.$$

Consider now the functions $f(x) = \sin(x)$ and $g(x) = x$.

(b) Qualitatively, which kinds of functions h will make $\|f - g\|_h$ big and which will make it small?

(c) Find a strictly positive continuous function h such that

$$\|f - g\|_h \leq \frac{1}{10} \quad \text{and} \quad \|h\|_1 = 1.$$

(There are, at least 2 ways to solve this problem. You can take an experimental approach and use your knowledge from part (b) and just guess at functions you know and see if they work, for this it is fine to use a computer program to actually compute integrals. A more pure math approach is to dream up a class of functions and prove that a function from that class will work. Both are completely valid methods.)

(d) Find a strictly positive continuous function h such that

$$\|f - g\|_h \geq 10 \quad \text{and} \quad \|h\|_1 = 1.$$