Liebnitz' Formula

1. For any $-1 < x < 1$ we have
$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}.$$

2. Define $f(x) = \frac{1}{1+x^2}$. Then f is continuous, so Lebesgue measurable, and positive so
$$\int_{(0,1)} f \, d\lambda$$
makes sense.

3. We will compute the integral in two ways to show
$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots .$$
As part of this we will show that
$$\int_{(0,1)} \sum_{n=0}^{\infty} (-1)^n x^{2n} \, d\lambda(x) = \sum_{n=0}^{\infty} (-1)^n \int_{(0,1)} x^{2n} \, d\lambda(x).$$
Note that neither the Monotone Convergence Theorem or the Dominated Convergence Theorem (in the form of Corollary 5.2) are immediately applicable.

The Γ-function

1. Let $\alpha \in \mathbb{R}$ be given and define $f_\alpha \colon (0,\infty) \to (0,\infty)$ by $f(x) = x^\alpha$. We will show
   a) $\int_{(0,1)} f_\alpha \, d\lambda < \infty$ if and only if $\alpha > -1$, and
   b) $\int_{(1,\infty)} f_\alpha \, d\lambda < \infty$ if and only if $\alpha < -1$.

2. Give $\alpha > -1$ define $g \colon (0,\infty) \to (0,\infty)$ by
$$g(x) = x^\alpha e^{-x}.$$
Then g is Lebesgue integrable over $(0,\infty)$.

3. Now define $\Gamma \colon (0,\infty) \to (0,\infty)$ by
$$\Gamma(t) = \int_{(0,\infty)} x^{t-1} e^{-x} \, d\lambda(x).$$
The above shows that Γ is well-defined. We will show that
$$\Gamma(t+1) = t\Gamma(t).$$
Since $\Gamma(1) = 1$ this shows that $\Gamma(n+1) = n!$ for all $n \in \mathbb{N}$.