Extra problem

Consider a measure space $(\Omega, \mathcal{A}, \mu)$. Let $\mathcal{D}$ be a sub-σ-algebra of $\mathcal{A}$, that is $\mathcal{D}$ is a σ-algebra of subsets of Ω and $\mathcal{D} \subseteq \mathcal{A}$, and let $\mu|_{\mathcal{D}}$ be the restriction of μ to $\mathcal{D}$. Suppose $f \colon \Omega \to \mathbb{C}$ is a $\mathcal{D}$-measurable function. Show:

1. f is $\mathcal{A}$-measurable.
2. If f is non-negative then
$$\int f \, d\mu|_{\mathcal{D}} = \int f \, d\mu.$$
3. $f \in \mathcal{L}(\Omega, \mathcal{D}, \mu|_{\mathcal{D}})$) if and only if $f \in \mathcal{L}(\Omega, \mathcal{A}, \mu)$.
4. If f is integrable then
$$\int f \, d\mu|_{\mathcal{D}} = \int f \, d\mu.$$

Note that this shows that if $f \colon \mathbb{R} \to \mathbb{C}$ is a Borel measurable function, then (in terms of integrating f) it does not matter if we consider the Lebesgue measure restricted to the Borel sets or to the measurable sets. Thus justifying that we so far have not bother to make a very clear distinction between the two.