Last time

1. We looked at inner product spaces: Linear spaces X with a inner product $\langle \cdot, \cdot \rangle \colon X \times X \to \mathbb{C}$.
2. We also introduced Hilbert spaces: Inner product spaces that are complete in the induced norm:
$$\|x\| = \langle x, x \rangle.$$
3. We prove basic properties of the inner product:
   3.1 $\langle x+y, x+y \rangle = \langle x, x \rangle + 2\Re\left(\langle x, y \rangle\right) + \langle y, y \rangle$.
   3.2 $|\langle x, y \rangle| \leq \|x\|\|y\|$ (Cauchy inequality).
   3.3 $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$ (parallelogram law).
   3.4 The inner product is continuous with respect to the norm topology.
4. Tried to show: Let K be a closed subspace of a Hilbert space H. For every point $x \in H$ there exists a point $y_0 \in K$ such that
$$\|x - y_0\| = \rho(x, K) \quad (:= \inf\{\|x-y\| \mid y \in K\}).$$

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1. We will show: Tried to show: Let K be a closed subspace of a Hilbert space H. For every point $x \in H$ there exists a unique point $y_0 \in K$ such that

$$\|x - y_0\| = \rho(x, K) \quad (:= \inf\{\|x - y\| \mid y \in K\}).$$

Furthermore y_0 is uniquely determined by the conditions

$$y_0 \in K \quad \text{and} \quad \langle x - y_0, y \rangle = 0 \text{ for all } y \in K.$$

2. We will introduce the notion of orthogonality and the orthogonal complement of a subset.
3. We will define the orthogonal projection onto a subspace (Exercise 13.26).
4. We will introduce the spaces $\mathcal{L}^p(\Omega, \mathcal{A}, \mu)$ and the spaces $L^p(\Omega, \mathcal{A}, \mu)$.
5. For $1 \leq p \leq \infty$, $L^p(\Omega, \mathcal{A}, \mu)$ is a Banach space. Moreover $L^2(\Omega, \mathcal{A}, \mu)$ is a Hilbert space.

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Existence of a nearest point

Proof.

- There exists a sequence $\{y_n\} \subseteq K$ such that

$$\lim_n \|x - y_n\| = \rho(x, K).$$

  We claim that $\{y_n\}$ is Cauchy.

- Apply the parallelogram law to $z_n = x - y_n$ and $w_m = x - y_m$ to get

$$\|z_n + w_m\|^2 + \|z_n - w_m\|^2 = 2\|z_n\|^2 + 2\|w_m\|^2.$$

- Rearrange terms and plug in what z_n and w_m are to get

$$\begin{aligned}\|y_m - y_n\|^2 &= 2\|x - y_n\|^2 + 2\|x - y_m\|^2 - 4\|x - \frac{1}{2}(y_n + y_m)\|^2 \\ &\leq 2\|x - y_n\|^2 + 2\|x - y_m\|^2 - 4\rho(x, K)^2 \to 0\end{aligned}$$

  as $n, m \to \infty$. Hence $\{y_n\}$ is Cauchy.

- The limit point y_0 of $\{y_n\}$ is in K and satisfies $\|x - y_0\| = \rho(x, K)$.

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