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1. We showed: Let K be a closed subspace of a Hilbert space H. For every point $x \in H$ there exists a unique point $y_0 \in K$ such that

$$\|x - y_0\| = \rho(x, K) \quad (:= \inf\{\|x - y\| \mid y \in K\}).$$

   Furthermore y_0 is uniquely determined by the conditions

$$y_0 \in K \quad \text{and} \quad \langle x - y_0, y \rangle = 0 \text{ for all } y \in K.$$

2. We introduced the notion of orthogonality and the orthogonal complement of a subset.
3. We defined the orthogonal projection P_K onto a subspace (Exercise 13.26) an showed basic properties:
   3.1 P_K is linear.
   3.2 $\|P_K x\| \|x\|$, for all $x \in H$.
   3.3 $P_K \circ P_K = P_K$.
4. Introduced the spaces $\mathcal{L}^p(\Omega, \mathcal{A}, \mu)$, $p \in [0, \infty]$

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Existence of a nearest point

Proof.

- There exists a sequence $\{y_n\} \subseteq K$ such that

$$\lim_n \|x - y_n\| = \rho(x, K).$$

  We claim that $\{y_n\}$ is Cauchy.
- Apply the parallelogram law to $z_n = x - y_n$ and $w_m = x - y_m$ to get

$$\|z_n + w_m\|^2 + \|z_n - w_m\|^2 = 2\|z_n\|^2 + 2\|w_m\|^2.$$

- Rearrange terms and plug in what z_n and w_m are to get

$$\begin{aligned}\|y_m - y_n\|^2 &= 2\|x - y_n\|^2 + 2\|x - y_m\|^2 - 4\|x - \frac{1}{2}(y_n + y_m)\|^2 \\ &\leq 2\|x - y_n\|^2 + 2\|x - y_m\|^2 - 4\rho(x, K)^2 \to 0\end{aligned}$$

  as $n, m \to \infty$. Hence $\{y_n\}$ is Cauchy.
- The limit point y_0 of $\{y_n\}$ is in K and satisfies $\|x - y_0\| = \rho(x, K)$.

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