Mandatory assignment in MAT3400/4400

October 6, 2016

The solution to the assignment must be delivered in the special box ("obligkassa") on the 7th floor in the Niels Henrik Abel building before *14:30 on Thursday the 27th of October, 2016*. Remember to fill in a cover page for the assignment before you put it in the box. Blank cover pages can be found near the "obligkassa". Students who take this course as MAT4400 must deliver the solution typeset in LaTeX whereas students who take the course as MAT3400 may deliver a hand-written solution.

You must justify all your answers.

The points in parentheses indicate the maximum score for each problem or subproblem. The maximum score is granted for a correct and complete solution to the respective question. The total number of points for a complete and correct assignment is 100 points. In order to have the assignment approved you must get a score of at least 45 points. If you do not achieve 45 points but have made a serious attempt at answering several questions in the assignment you will be given the chance to submit a revised version no later than Thursday the 10th of November, 2016.

Please read the information about mandatory assignments supplied by the department.

Norwegian: `http://www.uio.no/studier/admin/obligatoriske-aktiviteter/mn-math-oblig.html`

English: `http://www.uio.no/english/studies/admin/compulsory-activities/mn-math-mandatory.html`

Problem 1 (10 points)

Let $A = \{a_1, a_2, \ldots\}$ be a countable subset of $\mathbb{R}$ and let λ be the Lebesgue measure on $(\mathbb{R}, \mathcal{M})$. Define for each $k \in \mathbb{N}$

$$U_k = \bigcup_{n=1}^{\infty} \left(a_n - \frac{1}{2^{n+k}}, a_n + \frac{1}{2^{n+k}} \right).$$

Let

$$N = \bigcap_{m=1}^{\infty} U_m.$$

Show that $\lambda(N) = 0$.

Problem 2 (10 points)

Consider the measure space $(\mathbb{R}, \mathcal{M}, \lambda)$ and let $g = \chi_{[0,\infty)}$. Show that there is no continuous function f such that $f = g$ λ-a.e.

Problem 3 (15 points)

Consider the measure space $(\mathbb{R}, \mathcal{M}, \lambda)$ and define for each $n \in \mathbb{N}$ a function $f_n \colon \mathbb{R} \to \mathbb{R}$ by

$$f_n(x) = \frac{n \sin\left(\frac{x}{n}\right)}{x\,(1 + x^2)}.$$

Show that each f_n is integrable over $(0, \infty)$ and show that

$$\lim_n \int_{(0,\infty)} f_n(x)\, d\lambda = \frac{\pi}{2}.$$

Hint: For $t \in [0, \infty)$ we have $|\sin(t)| \leq t$.

Problem 4 (30 points)

Consider a measure space $(X, \mathcal{A}, \mu)$ and a measurable space $(Y, \mathcal{D})$. Let $f \colon X \to Y$ be a function such that $f^{-1}(D) \in \mathcal{A}$ for all $D \in \mathcal{D}$. Define a function $\mu_f \colon \mathcal{D} \to [0, \infty]$ by

$$\mu_f(D) = \mu(f^{-1}(D)).$$

We call μ_f the *image measure* of f.

(a) (10 points) Show that μ_f is indeed a measure on $(Y, \mathcal{D})$.

Let $g\colon Y \to \mathbb{C}$ be a $\mathcal{D}$-measurable function.

(b) (5 points) Show that $g \circ f$ is $\mathcal{A}$-measurable.

(c) (15 points) Show that if g is non-negative then

$$\int_Y g \, d\mu_f = \int_X g \circ f \, d\mu,$$

and that $g \in \mathcal{L}(Y, \mathcal{D}, \mu_f)$ if and only if $g \circ f \in \mathcal{L}(X, \mathcal{A}, \mu)$.
Hint: Use bootstraping. Start by showing that for any set $D \in \mathcal{D}$ we have $\chi_D \circ f = \chi_{f^{-1}(D)}$.

It can be shown – you are *not* asked to do this – that in the case g is integrable then

$$\int_Y g \, d\mu_f = \int_X g \circ f \, d\mu.$$

Problem 5 (15 points)

Consider the measure space $([0, 2\pi], \mathcal{B}_{[0,2\pi]}, \lambda)$. Let $\mathcal{B}_2$ be the Borel sets in $\mathbb{C}$, i.e. $\mathcal{B}_2$ is the smallest σ-algebra of $\mathbb{C}$ that contain all the open subsets of $\mathbb{C}$. Let $f\colon [0, 2\pi] \to \mathbb{C}$ be the function

$$f(\theta) = e^{i\theta}.$$

(a) (5 points) Show that for all $B \in \mathcal{B}_2$ we have $f^{-1}(B) \in \mathcal{B}_{[0,2\pi]}$.

Let λ_f be the image measure on $(\mathbb{C}, \mathcal{B}_2)$ defined in Problem 4.

(b) (10 points) Let $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$ and let $n \in \{1, 2, \ldots\}$. Show that

$$\int_{\mathbb{T}} z^n \, d\lambda_f = 0.$$

Hint: Use the results from Problem 4. You may do this even if you have not done Problem 4.

Problem 6 (20 points)

Consider a finite measure space $(\Omega, \mathcal{A}, \mu)$. Let $E \in \mathcal{A}$ be such that $\mu(E) \neq 0$ and $\mu(E^c) \neq 0$ and define a σ-algebra by $\mathcal{G} = \{\emptyset, E, E^c, \Omega\}$.

(a) (10 points) Show that a function $f\colon \Omega \to \mathbb{C}$ is $\mathcal{G}$ measurable if and only if

$$f = a\chi_E + b\chi_{E^c},$$

for some complex numbers a, b.

It can be shown – you are *not* asked to do this – that $L^2(\Omega, \mathcal{G})$ is a closed subspace of $L^2(\Omega, \mathcal{A})$. Let $P_{\mathcal{G}}$ be the orthogonal projection from $L^2(\Omega, \mathcal{A})$ onto $L^2(\Omega, \mathcal{G})$.

(b) (10 points) Show that for $f \in L^2(\Omega, \mathcal{A}, \mu)$

$$P_{\mathcal{G}}(f) = \alpha\chi_E + \beta\chi_{E^c},$$

where

$$\alpha = \frac{1}{\mu(E)} \int_E f \, d\mu, \quad \text{and} \quad \beta = \frac{1}{\mu(E^c)} \int_{E^c} f \, d\mu.$$