All references will be to McDonald and Weiss.

Problem 1

For $n, k \in \mathbb{N}$ we let
$$I_{n,k} = \left(a_n - \frac{1}{2^{n+k}}, a_n + \frac{1}{2^{n+k}}\right).$$
Since $I_{n,k}$ is an interval we have by Proposition 3.2 that
$$\lambda(I_{n,k}) = \frac{2}{2^{n+k}} = \frac{1}{2^n}\frac{1}{2^{k-1}}.$$
By subadditivity of the Lebesgue measure (Proposition 3.1) we then have
$$\lambda(U_k) = \lambda\left(\bigcup_{n=1}^{\infty} I_{n,k}\right) \leq \sum_{n=1}^{\infty} \frac{1}{2^n}\frac{1}{2^{k-1}} = \frac{1}{2^{k-1}}\sum_{n=1}^{\infty}\frac{1}{2^n} = \frac{1}{2^{k-1}},$$
for each $k \in \mathbb{N}$. Since $U_1 \supseteq U_2 \supseteq U_3 \cdots$ and $\lambda(U_1) < \infty$ it follows from downward continuity of measures (Theorem 5.1) that
$$\lambda(N) = \lambda\left(\bigcap_{k=1}^{\infty} U_k\right) = \lim_k \lambda(U_k) \leq \lim_k \frac{1}{2^{k-1}} = 0.$$
Hence $\lambda(N) = 0$.

Problem 2

Suppose for contradiction that $f\colon \mathbb{R} \to \mathbb{R}$ is a continuous function such that $f = g$ λ-a.e. For each $n \in \mathbb{N}$ we define intervals
$$E_n = \left(-\frac{1}{n}, 0\right) \quad \text{and} \quad F_n = \left(0, \frac{1}{n}\right).$$
Since E_n and F_n are intervals we have (Proposition 3.2) that $\lambda(E_n) = \frac{1}{n} = \lambda(F_n)$. In particular they do not have measure 0, so there exists points $x_n \in E_n$, $y_n \in F_n$ such that $f(x_n) = g(x_n) = 0$ and $f(y_n) = g(y_n) = 1$. For each n we have $|x_n| \leq \frac{1}{n}$ so $\lim_n x_n = 0$. Similarly $\lim_n y_n = 0$. Now by the continuity of f we get
$$f(0) = f(\lim_n x_n) = \lim_n f(x_n) = \lim_n 0 = 0,$$
but also
$$f(0) = f(\lim_n y_n) = \lim_n f(y_n) = \lim_n 1 = 1.$$
This is a clear contradiction, therefore there cannot exists a continuous function that is equal to g λ-a.e.

Problem 3 (15 points)

First note that all the f_n are continuous and therefore measurable. For each $n \in \mathbb{N}$ and $x \in (0, \infty)$ we have

$$|f_n(x)| = \left| \frac{n \sin\left(\frac{x}{n}\right)}{x\,(1+x^2)} \right| = \frac{n \left|\sin\left(\frac{x}{n}\right)\right|}{x\,(1+x^2)} \leq \frac{n\frac{x}{n}}{x\,(1+x^2)} = \frac{x}{x\,(1+x^2)} = \frac{1}{1+x^2}.$$

Define a continuous function $g\colon \mathbb{R} \to \mathbb{R}$ by

$$g(x) = \frac{1}{1+x^2}.$$

Then on the set $(0, \infty)$ we have $|f_n| \leq g$, so to show that each f_n is integrable it suffices to show that g is integrable over $(0, \infty)$. We put $E_n = [0, n]$ and note that $E_1 \subseteq E_2 \subseteq E_3 \subseteq \cdots$ and $\cup_n E_n = [0, \infty)$. Since g is a measurable non-negative function we get by the Monotone Convergence Theorem that

$$\int_{(0,\infty)} g\, d\lambda = \int_{[0,\infty)} g\, d\lambda = \lim_n \int_{[0,n]} g\, d\lambda.$$

Since the Riemann and the Lebesgue integrals agree for continuous functions over bounded intervals (Theorem 4.9), we then get

$$\begin{aligned}\int_{[0,n]} g\, d\lambda &= \int_0^n g(x)\, dx = \int_0^n \frac{1}{1+x^2}\, dx \\ &= [\arctan(x)]_0^n = \arctan(n) - \arctan(0) = \arctan(n).\end{aligned}$$

Thus

$$\int_{(0,\infty)} g\, d\lambda = \lim_n \int_{[0,n]} g\, d\lambda = \lim_n \arctan(n) = \frac{\pi}{2}.$$

In particular g is integrable and so each f_n is integrable.

We now note that for each $x \in (0, \infty)$ we have, by L'Hôpital's rule, that

$$\lim_n n \sin\left(\frac{x}{n}\right) = \lim_n \frac{\sin\left(\frac{x}{n}\right)}{\frac{1}{n}} = \lim_n \frac{x \cos\left(\frac{x}{n}\right)}{1} = x\cos(0) = x.$$

Hence for each $x \in (0, \infty)$ we have

$$\lim_n f_n(x) = \frac{\lim_n n \sin\left(\frac{x}{n}\right)}{x\,(1+x^2)} = \frac{x}{x\,(1+x^2)} = \frac{1}{(1+x^2)} = g(x).$$

It now follows from the Dominated Convergence Theorem (Theorem 5.9), using g as the dominating function, that

$$\lim_n \int_{(0,\infty)} f_n\, d\lambda = \int_{(0,\infty)} \lim_n f_n\, d\lambda = \int_{(0,\infty)} g\, d\lambda = \frac{\pi}{2}.$$

Problem 4

Part (a): We verify the conditions of Definition 5.1. First we see that for all $D \in \mathcal{D}$ we have

$$\mu_f(D) = \mu(f^{-1}(D)) \geq 0,$$

since μ is a measure. Since $f^{-1}(\emptyset) = \emptyset$ we see that

$$\mu_f(\emptyset) = \mu(f^{-1}(\emptyset)) = \mu(\emptyset) = 0.$$

Suppose now that $\{D_n\}_n$ is a sequence of disjoint sets from $\mathcal{D}$. Then $\{f^{-1}(D_n)\}_n$ is a collection of pairwise disjoint sets in $\mathcal{A}$, so

$$\begin{aligned}\mu_f\left(\bigcup_n D_n\right) &= \mu\left(f^{-1}\left(\bigcup_n D_n\right)\right) = \mu\left(\bigcup_n f^{-1}(D_n)\right) \\ &= \sum_n \mu(f^{-1}(D_n)) = \sum_n \mu_f(D_n),\end{aligned}$$

since μ is a measure.

Part (b): Since g is measurable we have that $g^{-1}(O) \in \mathcal{D}$ for any open subset O of $\mathbb{C}$. So by the definition of f we have that for any open set O

$$(g \circ f)^{-1}(O) = f^{-1}(g^{-1}(O)) \in \mathcal{A},$$

i.e., $g \circ f$ is $\mathcal{A}$-measurable.

Part (c): We will use bootstraping. We start by establishing the equality for indicator functions, then non-negative simple functions, and then non-negative functions. Let D be a set in $\mathcal{D}$. Then

$$\begin{aligned}(\chi_D \circ f)(x) &= \begin{cases} 1, & \text{if } f(x) \in D \\ 0, & \text{if } f(x) \notin D \end{cases} \\ &= \begin{cases} 1, & \text{if } x \in f^{-1}(D) \\ 0, & \text{if } x \notin f^{-1}(D) \end{cases} \\ &= \chi_{f^{-1}(D)}(x).\end{aligned}$$

Thus we have

$$\int_Y \chi_D \, d\mu_f = \mu_f(D) = \mu(f^{-1}(D)) = \int_X \chi_{f^{-1}(D)} \, d\mu = \int_X \chi_D \circ f \, d\mu.$$

So the two integrals agree on indicator functions. Let now $s\colon Y \to [0,\infty]$ be a simple function with canonical presentation

$$s = \sum_{i=1}^{n} a_i \chi_{A_i}.$$

Then

$$\begin{aligned}\int_Y s\,d\mu_f &= \int_Y \sum_{i=1}^n a_i\chi_{A_i}\,d\mu_f = \sum_{i=1}^n a_i \int_Y \chi_{A_i}\,d\mu \\ &= \sum_{i=1}^n a_i \int_X \chi_{A_i}\circ f\,d\mu = \int_X \sum_{i=1}^n a_i(\chi_{A_i}\circ f)\,d\mu \\ &= \int_X \left(\sum_{i=1}^n a_i\chi_{A_i}\right)\circ f\,d\mu = \int_X s\circ f\,d\mu.\end{aligned}$$

So we have established the equality for all non-negative simple functions. Suppose now $h\colon Y \to [0,\infty]$ is a non-negative $\mathcal{D}$-measurable function. By Proposition 5.7 we can find a non-decreasing sequence $\{s_n\}_n$ of non-negative simple $\mathcal{D}$-measurable functions such that s_n converges pointwise towards h. By the Monotone Convergence Theorem (Theorem 5.6) we then get

$$\int_Y h\,d\mu_f = \int_Y \lim_n s_n\,d\mu_f = \lim_n \int_Y s_n\,d\mu_f = \lim_n \int_X s_n\circ f\,d\mu.$$

Since $s_n\circ f$ is a non-decreasing sequence of $\mathcal{A}$-measurable functions that converges pointwise to $h\circ f$ we then get

$$\int_Y h\,d\mu_f = \lim_n \int_X s_n\circ f\,d\mu = \int_X \lim_n (s_n\circ f)\,d\mu = \int_X h\circ f\,d\mu.$$

So the equality holds for non-negative functions.

We can now show the claim about the integrability of g. We have

$$\begin{aligned} g\in\mathcal{L}(Y,\mathcal{D},\mu_f) &\iff \int_Y |g|\,d\mu_f<\infty \iff \int_X |g|\circ f\,d\mu<\infty \\ &\iff \int_X |g\circ f|\,d\mu<\infty \iff g\circ f\in\mathcal{L}(X,\mathcal{A},\mu).\end{aligned}$$

Problem 5

Part (a): Since f is continuous it is measurable. It now follows from Exercise 5.30 that $f^{-1}(B)\in\mathcal{B}_{[0,2\pi]}$ for all $B\in\mathcal{B}_2$.

Part (b): For each $n\in\{1,2,\dots\}$ we have

$$\int_{\mathbb{T}} |z^n|\,d\lambda_f = \int_{\mathbb{T}} 1\,d\lambda_f = \lambda_f(\mathbb{T}) = \lambda(f^{-1}(\mathbb{T})) = \lambda([0,2\pi]) = 2\pi<\infty.$$

So each function z^n is integrable.

By the remarks after Problem 4 we then have

$$\int_{\mathbb{T}} z^n \, d\lambda_f = \int_{[0,2\pi]} f(\theta)^n \, d\lambda(\theta) = \int_{[0,2\pi]} \exp(in\theta) \, d\lambda(\theta)$$

Since $\exp(in\theta) = i\sin(n\theta) + \cos(n\theta)$, we then get that

$$\begin{aligned}\int_{\mathbb{T}} z^n \, d\lambda_f &= \int_{[0,2\pi]} \cos(n\theta) \, d\lambda(\theta) + i \int_{[0,2\pi]} \sin(n\theta) \, d\lambda(\theta) \\ &= \int_0^{2\pi} \cos(n\theta) \, d\theta + i \int_0^{2\pi} \sin(n\theta) \, d\theta \\ &= [\frac{1}{n} \sin(n\theta)]_0^{2\pi} + i[-\frac{1}{n} \cos(n\theta)]_0^{2\pi} \\ &= 0 + i0 = 0.\end{aligned}$$

Where we used that the Riemann and Lebesgue integrals agree for continuous functions over closed bounded intervals (Theorem 4.9).

Problem 6 (20 points)

Part (a): Suppose first that we are given a function of the form

$$f = a\chi_E + b\chi_{E^c},$$

with $a, b \in \mathbb{C}$. The indicator function of a set in $\mathcal{G}$ is $\mathcal{G}$-measurable, so the functions χ_E and χ_{E^c} are $\mathcal{G}$-measurable. Since the $\mathcal{G}$-measurable functions form an algebra (Theorem 5.3 and the remark after Example 5.4), we have that f is $\mathcal{G}$-measurable.

Suppose now that f is any $\mathcal{G}$-measurable function. We first claim that f can take at most two distinct values. Suppose that $a_1, a_2, a_3 \in \mathbb{C}$ are distinct and that $a_i \in f(\Omega)$ for $i = 1, 2, 3$. Each of the one-point sets $\{a_i\}$ is closed, so $f^{-1}(\{a_i\}) \in \mathcal{G}$ are three distinct sets in $\mathcal{G}$. Neither is the empty set, since we assumed $a_i \in f(\Omega)$, and neither is Ω, since we assumed that f took more than one value. There are only two sets in $\mathcal{G}$ not equal to $\emptyset$ or Ω, hence we have a contradiction. Since f only takes at most two values it is a simple $\mathcal{G}$-measurable function and so has the form

$$f = a\chi_\emptyset + b\chi_E + c\chi_{E^c} + d\chi_\Omega = (b + d)\chi_E + (c + d)\chi_{E^c},$$

for some $a, b, c, d \in \mathbb{C}$.

Part (b): By part (a) we know that $P_{\mathcal{G}} f = \alpha\chi_E + \beta\chi_{E^c}$ for some complex numbers $\alpha, \beta \in \mathbb{C}$. We wish to determine α and β. Let now $a, b \in \mathbb{C}$ be given. For any $\mathcal{A}$-measurable function g we get

$$\begin{aligned}\langle g, a\chi_E + b\chi_{E^c}\rangle &= \int_\Omega g\overline{a\chi_E + b\chi_{E^c}} \, d\mu = \overline{a} \int_\Omega g\chi_E \, d\mu + \overline{b} \int_\Omega g\chi_{E^c} \, d\mu \\ &= \overline{a} \int_E g \, d\mu + \overline{b} \int_E g \, d\mu.\end{aligned}$$

If we put $g = f - (\alpha\chi_E + \beta\chi_{E^c})$ then

$$\int_E g\,d\mu = \int_E f\,d\mu - \int_E \alpha\chi_E\,d\mu - \int_E \beta\chi_{E^c}\,d\mu = \int_E f\,d\mu - \alpha\mu(E),$$

where we used that $\chi_{E^c} = 0$ on E and $\chi_E = 1$ on E. Similarly

$$\int_{E^c} g\,d\mu = \int_{E^c} f\,d\mu - \int_{E^c} \alpha\chi_E\,d\mu - \int_{E^c} \beta\chi_{E^c}\,d\mu = \int_{E^c} f\,d\mu - \beta\mu(E^c),$$

So we get that

$$\begin{aligned}\langle f - (\alpha\chi_E + \beta\chi_{E^c}), a\chi_E + b\chi_{E^c}\rangle &= \langle g, a\chi_E + b\chi_{E^c}\rangle \\ &= \overline{a}\left(\int_E f\,d\mu - \alpha\mu(E)\right) + \overline{b}\left(\int_{E^c} f\,d\mu - \beta\mu(E^c)\right)\end{aligned}$$

By Theorem 13.2 we must have that the above inner product is 0 for all a, b in $\mathbb{C}$ (and that this uniquely identifies α and β). If we put $a = 1$ and $b = 0$ we get

$$0 = \langle f - (\alpha\chi_E + \beta\chi_{E^c}), \chi_E\rangle = \int_E f\,d\mu - \alpha\mu(E).$$

Hence

$$\alpha = \frac{1}{\mu(E)}\int_E f\,d\mu.$$

If we put $b = 1$ and $a = 0$ we see that

$$\beta = \frac{1}{\mu(E^c)}\int_{E^c} f\,d\mu.$$