

UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in: MAT3400/4400 — Linear analysis with applications.

Day of examination: Friday, December 5, 2014.

Examination hours: 9.00–13.00.

This problem set consists of 3 pages.

Appendices: None.

Permitted aids: None.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Note: You must justify all your answers!

Problem 1

Let $(\Omega, \mathcal{A}, \mu)$ be a measure space. For A in \mathcal{A} we let χ_A denote the characteristic function of A . For $1 \leq p < \infty$ let $\mathcal{L}^p(\Omega, \mathcal{A}, \mu)$ denote the space of complex-valued, \mathcal{A} -measurable functions g on Ω such that $\int_{\Omega} |g|^p d\mu < \infty$.

1a

(15 points) Let $\{f_n\}_{n \in \mathbb{N}}$ be a sequence of \mathcal{A} -measurable functions on Ω with values in $[0, \infty]$ for every $n \in \mathbb{N}$. Use the monotone convergence theorem to show that

$$\int_{\Omega} \left(\sum_{n=1}^{\infty} f_n \right) d\mu = \sum_{n=1}^{\infty} \left(\int_{\Omega} f_n d\mu \right).$$

1b

(10 points) Assume that there exists a sequence of pairwise disjoint sets $\{A_n\}_{n \in \mathbb{N}}$ in \mathcal{A} such that $1 \leq \mu(A_n) < \infty$ for all $n \in \mathbb{N}$. Let

$$f = \sum_{n=1}^{\infty} \frac{1}{n} \mu(A_n)^{-1} \chi_{A_n}.$$

Show that $f \notin \mathcal{L}^1(\Omega, \mathcal{A}, \mu)$ and that $f \in \mathcal{L}^2(\Omega, \mathcal{A}, \mu)$. Give an example of a measure space $(\Omega, \mathcal{A}, \mu)$ and sets $\{A_n\}_{n \in \mathbb{N}}$ where the hypotheses are satisfied.

1c

(10 points) Assume that $\mu(\Omega) < \infty$. Use the fact that $1 \in \mathcal{L}^2(\Omega, \mathcal{A}, \mu)$ to show that $\mathcal{L}^2(\Omega, \mathcal{A}, \mu) \subseteq \mathcal{L}^1(\Omega, \mathcal{A}, \mu)$.

(Continued on page 2.)

Problem 2

In this problem the following are assumed known: if $(\Omega, \mathcal{A}, \mu)$ is a measure space, then $L^2(\mu)$ is the Hilbert space of complex-valued, \mathcal{A} -measurable, square-integrable functions on Ω (where functions that are equal μ -almost everywhere are identified) with inner product given by

$$(g_1, g_2) = \int_{\Omega} g_1(x) \overline{g_2(x)} d\mu(x)$$

for $g_1, g_2 \in L^2(\mu)$. A sequence $\{e_k\}_{k \in \mathbb{N}}$ is an orthonormal basis for $L^2(\mu)$ if and only if Parseval's identity $\|g\|_2^2 = \sum_{k \in \mathbb{N}} |(g, e_k)|^2$ is valid for each $g \in L^2(\mu)$.

2a

(10 points) Consider the measure space $[0, 1]$ with the Borel σ -algebra $\mathcal{B}_{[0,1]}$ and the Lebesgue measure λ restricted to $\mathcal{B}_{[0,1]}$. Suppose that $\{u_m\}_{m \in \mathbb{N}}$ and $\{v_n\}_{n \in \mathbb{N}}$ are two orthonormal bases in $L^2(\lambda)$. Show that the functions $\{w_{n,m}\}_{n,m \in \mathbb{N}}$ defined by $w_{n,m}(x, y) = u_m(x)v_n(y)$ for $(x, y) \in [0, 1] \times [0, 1]$ form an orthonormal family in $L^2(\lambda \times \lambda)$ where $\lambda \times \lambda$ is the product measure on $\mathcal{B}_{[0,1]} \otimes \mathcal{B}_{[0,1]}$.

2b

(10 points) Let $f \in L^2(\lambda \times \lambda)$. Show that for each $n \in \mathbb{N}$, the function

$$F_n(x) = \int_{[0,1]} f(x, y) \overline{v_n(y)} d\lambda(y)$$

is in $L^2(\lambda)$ for λ -almost all $x \in [0, 1]$. Show that $(f, w_{n,m})$, computed as an inner product in $L^2(\lambda \times \lambda)$, is equal to (F_n, u_m) , computed as an inner product in $L^2(\lambda)$, for every $n, m \in \mathbb{N}$. (Hint: Use the Cauchy-Schwarz inequality and Fubini's theorem. To apply Fubini's theorem, refer to Problem 1c).

2c

(10 points) Conclude that $\{w_{n,m}\}_{n,m \in \mathbb{N}}$ forms an orthonormal basis for $L^2(\lambda \times \lambda)$. (Hint: Use Parseval's identity for each of the bases $\{u_m\}_m$ and $\{v_n\}_n$ and Problem 1a).

Problem 3

Let H be a Hilbert space over \mathbb{C} with inner product (\cdot, \cdot) and let $B(H)$ denote the space of bounded linear operators from H to H . Let $I \in B(H)$ denote the identity operator $I(x) = x$ for all $x \in H$.

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An operator $R \in B(H)$ is called an orthogonal reflection if there exists a closed subspace M of H such that

$$x + R(x) \in M \text{ and } x - R(x) \in M^\perp \text{ for every } x \in H.$$

3a

(15 points) Prove that if R is an orthogonal reflection, then $\frac{1}{2}(R + I)$ is an orthogonal projection.

3b

(10 points) Prove conversely that if $R \in B(H)$ is such that $\frac{1}{2}(R + I)$ is an orthogonal projection in $B(H)$, then R is an orthogonal reflection.

3c

(10 points) Prove that R is an orthogonal reflection if and only if $R = R^*$ and $R^2 = I$.

END