MAT3400/MAT4400

Mandatory assignment 1 of 1

Submission deadline
Monday 23rd April 2018, 14:30 at Devilry (https://devilry.ifi.uio.no).

Instructions
You can choose between scanning handwritten notes or typing the solution directly on a computer (for instance with LaTeX). The assignment must be submitted as a single PDF file. Scanned pages must be clearly legible. The submission must contain your name, course and assignment number.

It is expected that you give a clear presentation with all necessary explanations. Remember to include all relevant plots and figures. Students who fail the assignment, but have made a genuine effort at solving the exercises, are given a second attempt at revising their answers. All aids, including collaboration, are allowed, but the submission must be written by you and reflect your understanding of the subject. If we doubt that you have understood the content you have handed in, we may request that you give an oral account.

In exercises where you are asked to write a computer program, you need to hand in the code along with the rest of the assignment. It is important that the submitted program contains a trial run, so that it is easy to see the result of the code.

Application for postponed delivery
If you need to apply for a postponement of the submission deadline due to illness or other reasons, you have to contact the Student Administration at the Department of Mathematics (e-mail: studieinfo@math.uio.no) well before the deadline.

All mandatory assignments in this course must be approved in the same semester, before you are allowed to take the final examination.

Complete guidelines about delivery of mandatory assignments:
ui.no/english/studies/admin/compulsory-activities/mn-math-mandatory.html

GOOD LUCK!
Students who take this course as MAT4400 must deliver the solution typeset in \LaTeX{}, students who take the course as MAT3400 may deliver a (scan of a) hand-written solution.

The points in parentheses indicate the maximum score for each problem or subproblem. The maximum score is granted for a correct and complete solution to the respective question. The total number of points for a complete and correct assignment is 100 points. In order to have the assignment approved you must get a score of at least 45 points.

**Problem 1** (10 points). Let $M$ denote the $\sigma$-algebra of Lebesgue measurable subsets of $[0, \infty)$ and let $\lambda$ denote the Lebesgue measure on $([0, \infty), M)$. Recall that for an $x \in [0, \infty)$ we denote by $\lfloor x \rfloor$ the floor of $x$, i.e. $x$ rounded down to an integer.

Define a function $f: [0, \infty) \to \mathbb{R}$ by

$$f(x) = \exp(-\lfloor x \rfloor).$$

Show that $f$ is $M$-measurable and compute

$$\int_{[0, \infty)} f \, d\lambda.$$

**Problem 2** (10 points). Define a function $\mu: \mathcal{P}(\mathbb{R}) \to [0, \infty)$ by

$$\mu(A) = \sum_{n \in \mathbb{N} \cap A} \frac{1}{n^3},$$

where $\mathbb{N} = \{1, 2, 3, \ldots\}$ denotes the set of natural numbers. By Example 5.7 in M&W (p163) $\mu$ is a measure.

Consider the functions $g, h: \mathbb{R} \to \mathbb{R}$ given by

$$g(x) = \sin(x), \quad h(x) = x.$$

Determine for which $p \in [1, \infty)$ we have $g \in L^p(\mathbb{R}, \mathcal{P}(\mathbb{R}), \mu)$ and for which we have $h \in L^p(\mathbb{R}, \mathcal{P}(\mathbb{R}), \mu)$.

**Problem 3** (30 points). Let $(\Omega, \mathcal{A}, \mu)$ be a measure space, and let $p \in [1, \infty)$. Suppose we are given a sequence $\{f_n\}_n \subseteq L^p(\mu)$ and a function $f \in L^p(\mu)$, such that $\{f_n\}_n$ converges pointwise to $f$ $\mu$-almost everywhere.

(a) (10 points) Show that if $f_n$ converges to $f$ w.r.t. $\| \cdot \|_p$, then $\|f_n\|_p \to \|f\|_p$ as $n \to \infty$.

(b) (20 points) Conversely, show that if $\|f_n\|_p \to \|f\|_p$ as $n \to \infty$, then $f_n$ converges to $f$ w.r.t. $\| \cdot \|_p$.

**HINT:** Modify the proof of the Dominated Convergence Theorem.
Problem 4 (50 points). Denote by \( \mathcal{B} \) the Borel sets on \( \mathbb{R} \). A measure \( \mu \) on \( (\mathbb{R}, \mathcal{B}) \) is said to be finite if \( \mu(\mathbb{R}) < \infty \). We denote by \( \text{FM} \) the collection of finite measures on \( (\mathbb{R}, \mathcal{B}) \).

We denote by \( C_b(\mathbb{R}) \) the set of bounded continuous functions from \( \mathbb{R} \) to \( \mathbb{R} \), and by \( C_0(\mathbb{R}) \) the set of continuous functions that tend to 0 at \( \pm \infty \). In symbols
\[
C_b(\mathbb{R}) = \left\{ f : \mathbb{R} \to \mathbb{R} \mid f \text{ is continuous, } \sup \{|f(x)| \mid x \in \mathbb{R}\} < \infty \right\},
\]
and
\[
C_0(\mathbb{R}) = \left\{ f : \mathbb{R} \to \mathbb{R} \mid f \text{ is continuous, } \lim_{x \to \infty} f(x) = 0, \lim_{x \to -\infty} f(x) = 0 \right\}.
\]

Note that \( C_0(\mathbb{R}) \subseteq C_b(\mathbb{R}) \).

(a) (10 points) Show that if \( f \in C_b(\mathbb{R}) \) and \( \mu \in \text{FM} \) then \( f \) is \( \mu \)-integrable, i.e. \( f \in L^1(\mathbb{R}, \mathcal{B}, \mu) \).

If \( \{\mu_n\}_n \subseteq \text{FM} \) is a sequence of measures and \( \mu \in \text{FM} \), we say that \( \mu_n \) converges vaguely to \( \mu \), written \( \mu_n \vrightarrow \mu \), if
\[
\int f \, d\mu_n \rightarrow \int f \, d\mu \text{ as } n \rightarrow \infty
\]
for all \( f \in C_0(\mathbb{R}) \).

(b) (10 points) Let a sequence \( \{\mu_n\}_n \subseteq \text{FM} \) be given. Show that if \( \mu, \nu \in \text{FM} \) and \( \mu_n \vrightarrow \mu \) converges vaguely to both \( \mu \) and \( \nu \), then \( \mu = \nu \).

Suppose from now on that we are given a sequence \( \{\mu_n\}_n \subseteq \text{FM} \) and \( \mu \in \text{FM} \) such that
\[
\mu_n \vrightarrow \mu,
\]
and
\[
\mu_n(\mathbb{R}) = 1 = \mu(\mathbb{R}), \quad \text{for all } n \in \mathbb{N}.
\]

For each \( K \in \mathbb{N} \) define a function \( \phi_K \in C_0(\mathbb{R}) \) by
\[
\phi_K(x) = \begin{cases} 
0, & x \leq -K - 1; \\
x + K + 1, & -K - 1 < x < -K; \\
1, & -K \leq x \leq K; \\
-x + K + 1, & K < x < K + 1; \\
0, & K \leq x.
\end{cases}
\]

Note that each \( \phi_K \) is a continuous function taking the value 1 on \([-K, K]\) and 0 outside of \([-K - 1, K + 1]\). See Figure 1 for the graphs of some \( \phi_K \).
(c) (10 points) Let $\varepsilon > 0$ be given. Show that there exists $K \in \mathbb{N}$ such that
$$\left| \int_{\mathbb{R}} (1 - \phi_K) \, d\mu \right| \leq \varepsilon$$

(d) (20 points) Show that for all $f \in C_b(\mathbb{R})$ we have
$$\int_{\mathbb{R}} f \, d\mu_n \to \int_{\mathbb{R}} f \, d\mu \text{ as } n \to \infty.$$ 

**HINT:** First show that for all $n, K \in \mathbb{N}$ we have:
$$\left| \int_{\mathbb{R}} f \, d\mu_n - \int_{\mathbb{R}} f \, d\mu \right| \leq \left| \int_{\mathbb{R}} f \, d\mu_n - \int_{\mathbb{R}} f \phi_K \, d\mu_n \right| + \left| \int_{\mathbb{R}} f \phi_K \, d\mu_n - \int_{\mathbb{R}} f \phi_K \, d\mu \right| + \left| \int_{\mathbb{R}} f \phi_K \, d\mu - \int_{\mathbb{R}} f \, d\mu \right|$$

Then show that for all $\varepsilon > 0$ you can pick $N, K \in \mathbb{N}$ such that each term is less than $\frac{\varepsilon}{3}$ whenever $n \geq N$. It is important that you use the same $K$ for all three estimates.

**NOTE:** Showing that a norm difference $\|x_n - z_n\|$ tends to 0 by approximating $x_n$ with some $x'_n$ and $z_n$ with some $z'_n$ such that $\|x'_n - z'_n\| \to 0$, to get
$$\|x_n - z_n\| \leq \|x_n - x'_n\| + \|x'_n - z'_n\| + \|z'_n - z_n\|,$$

is a common technique in analysis. It is often called an $\frac{\varepsilon}{3}$-argument.