UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: MAT3400/4400 — Linear analysis with applications

Day of examination: June 8, 2023

Examination hours: 15.00 – 19.00

This problem set consists of 7 pages.

Appendices: None.

Permitted aids: None.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

The points in parentheses indicate the maximum possible score for each problem or subproblem. If you are unable to solve a subproblem, you may assume the result of that problem when solving later problems. E.g., if you cannot solve problem 3b, you may assume the result of 3b and try to solve 3c.

Note: You must justify all your answers!

Problem 1 (weight 15 points)

Let $(X, \mathcal{A}, \mu)$ be a measure space which is σ-finite, that is, there exists a sequence $\{A_k\}_{k\in\mathbb{N}}$ in $\mathcal{A}$ such that $X = \bigcup_{k=1}^{\infty} A_k$ and $\mu(A_k) < \infty$ for every $k \in \mathbb{N}$.

1a (weight 5 points)

Show that there exists a sequence $\{B_n\}_{n\in\mathbb{N}}$ in $\mathcal{A}$ satisfying that $B_n \subseteq B_{n+1}$ and $\mu(B_n) < \infty$ for every $n \in \mathbb{N}$, and that $X = \bigcup_{n=1}^{\infty} B_n$.

Solution. For each $n \in \mathbb{N}$, set $B_n := \bigcup_{k=1}^{n} A_k \in \mathcal{A}$.

Then $B_n \subseteq B_n \cup A_{n+1} = \bigcup_{k=1}^{n+1} A_k = B_{n+1}$ and $\mu(B_n) \leq \sum_{k=1}^{n} \mu(A_k) < \infty$ for every n. Moreover, since $A_n \subseteq B_n$ for each n, we have

$$X = \bigcup_{n=1}^{\infty} A_n \subseteq \bigcup_{n=1}^{\infty} B_n \subseteq X,$$

hence $X = \bigcup_{n=1}^{\infty} B_n$.

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1b (weight 10 points)

Show that for each $E \in \mathcal{A}$, we have

$$\mu(E) = \sup\big\{\mu(F) \mid F \in \mathcal{A},\, F \subseteq E,\, \mu(F) < \infty\big\}.$$

Solution. Let $E \in \mathcal{A}$ and set $S := \sup\big\{\mu(F) \mid F \in \mathcal{A},\, F \subseteq E,\, \mu(F) < \infty\big\}$. If $F \in \mathcal{A}$ and $F \subseteq E$, then $\mu(F) \leq \mu(E)$, so it is clear that $S \leq \mu(E)$.

If $\mu(E) < \infty$, then as $E \subseteq E$, we also get that $\mu(E) \leq S$, hence that $\mu(E) = S$.

Assume now that $\mu(E) = \infty$. To show that $\mu(E) = S$, we have then to show that $S = \infty$. For each $n \in \mathbb{N}$, set $E_n := E \cap B_n \in \mathcal{A}$. Then, using 1a, we get that $E_n \subseteq E_{n+1}$ and $\mu(E_n) \leq \mu(B_n) < \infty$ for every $n \in \mathbb{N}$. Further,

$$E = X \cap E = \bigcup_{n=1}^{\infty}(B_n \cap E) = \bigcup_{n=1}^{\infty} E_n.$$

By continuity from below for μ, we get that

$$\lim_{n\to\infty} \mu(E_n) = \mu(E) = \infty.$$

But, as $E_n \subseteq E$, we also have that $\mu(E_n) \leq S$. Letting $n \to \infty$, we get $S = \infty$, as desired.

Problem 2 (weight 35 points)

In this problem we consider the measure space $(X, \mathcal{A}, \mu)$, where $X = [0, \infty)$, $\mathcal{A}$ denotes the σ-algebra of all Lebesgue measurable subsets of $[0, \infty)$, and μ denotes the Lebesgue measure on $\mathcal{A}$.

2a (weight 10 points)

Let $g : X \to \mathbb{R}$ be the nonnegative Lebesgue measurable function defined by

$$g(x) = |x - 2|\, e^{-x} \quad \text{for all } \ x \in X.$$

Compute the integral $\int_X g\, d\mu$.

(You don't have to explain why g is Lebesgue measurable. You can freely use that $\int (x-2)e^{-x}\, dx = (1-x)e^{-x} + C$.)

Solution. Since $\{g\,\mathbf{1}_{[0,n]}\}_{n\in\mathbb{N}}$ is an increasing sequence of nonnegative Lebesgue measurable functions converging pointwise to g on X, the MCT gives that

$$\int_X g\, d\mu = \lim_{n\to\infty} \int_X g\,\mathbf{1}_{[0,n]}\, d\mu = \lim_{n\to\infty} \int_{[0,n]} g\, d\mu\,.$$

(Continued on page 3.)

Let now $n > 2$. Since g is continuous, hence Riemann-integrable, on $[0, n]$, we have that

$$\int_{[0,n]} g\, d\mu = \int_0^n g(x)\, dx = \int_0^2 (2-x)\, e^{-x}\, dx + \int_2^n (x-2)\, e^{-x}\, dx$$

$$= \big[(x-1)e^{-x}\big]_0^2 + \big[(1-x)e^{-x}\big]_2^n = e^{-2} + 1 + (1-n)e^{-n} + e^{-2}\,.$$

Since $(1-n)e^{-n} \to 0$ as $n \to \infty$, we get that

$$\int_X g\, d\mu = \lim_{n\to\infty} (1 + 2\, e^{-2} + (1-n)e^{-n}) = 1 + 2\, e^{-2}\,.$$

2b (weight 15 points)

For each $n \in \mathbb{N}$, let $f_n : X \to \mathbb{R}$ be the Lebesgue measurable function defined by

$$f_n(x) = \frac{n\,(x-2)\, e^{-x}}{2n + \sin x} \quad \text{for all} \quad x \in X.$$

Show that each f_n is integrable w.r.t. μ. Show also that the limit

$$\lim_{n\to\infty} \int_X f_n\, d\mu$$

exists and find its value. (You don't have to explain why each f_n is Lebesgue measurable.)

Solution. Let $n \in \mathbb{N}$ and $x \geq 0$. Since $1 \leq (2 + \frac{1}{n}\sin x) \leq 3$, we have that

$$\left|\frac{n}{2n + \sin x}\right| = \frac{1}{2 + \frac{1}{n}\sin x} \leq 1.$$

Thus,

$$\big|f_n(x)\big| = \left|\frac{n}{2n + \sin x}\right| |x-2|\, e^{-x} \leq |x-2|\, e^{-x} = g(x)$$

for all $n \in \mathbb{N}$ and all $x \in X$, i.e., $|f_n| \leq g$ on X for all $n \in \mathbb{N}$.

Let $f : X \to \mathbb{R}$ be given by $f(x) = \frac{1}{2}(x-2)e^{-x}$. Since

$$\frac{n}{2n + \sin x} = \frac{1}{2 + \frac{1}{n}\sin x} \to \frac{1}{2} \quad \text{as } n \to \infty$$

for every $x \in X$, we get that $\{f_n\}_{n\in\mathbb{N}}$ converges pointwise to f on X.

Now, since g is integrable on X by 2a, we can apply the LDCT and get that each f_n is integrable on X, f is integrable on X, and

$$\lim_{n\to\infty} \int_X f_n\, d\mu = \int_X f\, d\mu\,.$$

(Continued on page 4.)

Finally, we have to compute $\int_X f \, d\mu$. Since $|f\mathbf{1}_{[0,n]}| \le |f|$ on X for every $n \in \mathbb{N}$, and $|f| = g$ is integrable on X, the LDCT gives that $f\mathbf{1}_{[0,n]}$ is integrable for every $n \in \mathbb{N}$ and

$$\int_X f \, d\mu = \lim_{n\to\infty} \int_X f\mathbf{1}_{[0,n]} \, d\mu = \lim_{n\to\infty} \int_{[0,n]} f \, d\mu = \lim_{n\to\infty} \int_0^n \frac{1}{2}\,(x-2)e^{-x} \, dx$$

$$= \lim_{n\to\infty} \frac{1}{2}\left[(1-x)e^{-x}\right]_0^n = \lim_{n\to\infty} \frac{1}{2}\left((1-n)e^{-n} - 1\right) = -\frac{1}{2}\,,$$

where we have used that f is Riemann-integrable on $[0,n]$ for every $n \in \mathbb{N}$.

2c (weight 10 points)

Consider now the Hilbert space $H = L^2(X, \mathcal{A}, \mu)$. To simplify notation, we consider elements of H as complex functions on X, i.e., we identify complex measurable functions on X which agree μ-almost everywhere.

For each $k \in \mathbb{N}$, set $f_k := \mathbf{1}_{[k-1,k)}$ and note that $f_k \in H$ (you can take this as granted). Let M be the closed subspace of H given by

$$M = \overline{\operatorname{span}\{f_k \mid k \in \mathbb{N}\}},$$

and let P_M denote the orthogonal projection of H on M.

Let $f \in H$. Explain why f is integrable over each interval $[k-1,k)$. Then set $h := P_M(f)$ and $c_k := \int_{[k-1,k)} f \, d\mu$ for each $k \in \mathbb{N}$. Show that

$$\Big\|h - \sum_{k=1}^n c_k\, f_k\Big\|_2 \to 0 \quad \text{as } n \to \infty\,.$$

Solution. Since $f \in H$, f and $|f|$ are Lebesgue measurable, and $|f| \in H$. For each $k \in \mathbb{N}$, Hölder's inequality gives that

$$\int_{[k-1,k)} |f| \, d\mu = \int_X |f|\, f_k \, d\mu \le \|f\|_2\, \|f_k\|_2 < \infty,$$

hence that f is integrable over $[k-1,k)$.

For $k, l \in \mathbb{N}$, we have

$$\langle f_k, f_l\rangle = \int_X \mathbf{1}_{[k-1,k)}\overline{\mathbf{1}_{[l-1,l)}} \, d\mu = \int_X \mathbf{1}_{[k-1,k)\cap[l-1,l)} \, d\mu = \begin{cases} \mu([k-1,k)) = 1 & \text{if } k = l, \\ \mu(\emptyset) = 0 & \text{if } k \ne l\,. \end{cases}$$

Thus $\{f_k\}_{k\in\mathbb{N}}$ is orthonormal. By definition of M, it follows that $\{f_k\}_{k\in\mathbb{N}}$ is an orthonormal basis for M. We therefore get that

$$h = P_M(f) = \sum_{k=1}^\infty \langle f, f_k\rangle\, f_k \quad \text{(convergence w.r.t. } \|\cdot\|_2).$$

Since $\langle f, f_k\rangle = \int_X f\,\overline{f_k} \, d\mu = \int_{[k-1,k)} f \, d\mu = c_k$ for each $k \in \mathbb{N}$, this means precisely that

$$\Big\|h - \sum_{k=1}^n c_k\, f_k\Big\|_2 \to 0 \quad \text{as } n \to \infty.$$

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Problem 3 (weight 30 points)

Let H be a Hilbert space (over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$) having a countably infinite orthonormal basis $\{u_n\}_{n\in\mathbb{N}}$.

A sequence $\{v_n\}_{n\in\mathbb{N}}$ of vectors in H is called a *Bessel sequence* if there exists a constant $M > 0$ such that

$$\sum_{n=1}^{\infty} |\langle x, v_n\rangle|^2 \le M\,\|x\|^2 \quad \text{for all } x \in H.$$

3a (weight 10 points)

Let $T \in \mathcal{B}(H)$, $T \neq 0$, and set $v_n := T(u_n)$ for each $n \in \mathbb{N}$. Use Parseval's identity in a suitable way to show that $\{v_n\}_{n\in\mathbb{N}}$ is a Bessel sequence.

Solution. Using Parseval's identity to get the the third equality, we get that

$$\sum_{n=1}^{\infty} |\langle x, v_n\rangle|^2 = \sum_{n=1}^{\infty} |\langle x, T(u_n)\rangle|^2 = \sum_{n=1}^{\infty} |\langle T^*(x), u_n\rangle|^2 = \|T^*(x)\|^2 \le \|T^*\|^2\,\|x\|^2$$

for every $x \in H$. Setting $M := \|T^*\|^2$ $(= \|T\|^2 > 0)$, we see that $\{v_n\}_{n\in\mathbb{N}}$ is a Bessel sequence.

Let now $\{v_n\}_{n\in\mathbb{N}}$ be a Bessel sequence in H. Let H_0 denote the subspace of H given by $H_0 = \operatorname{span}\{u_n \mid n \in \mathbb{N}\}$, and define a linear map $T_0 : H_0 \to H$ by

$$T_0\Big(\sum_{n=1}^{N} c_n\, u_n\Big) = \sum_{n=1}^{N} c_n\, v_n$$

whenever $N \in \mathbb{N}$ and $c_1, \ldots, c_N \in \mathbb{F}$. (Note that if $x_0 \in H_0$, then $x_0 \in \operatorname{span}\{u_1, \ldots, u_N\}$ for some $N \in \mathbb{N}$, so this definition makes sense.)

3b (weight 10 points)

Show that T_0 is bounded.

Hint. To estimate $\|T_0(x_0)\|$ for $x_0 \in H_0$, a good start is to use that $\|y\| = \sup_{x\in H, \|x\|\le 1} |\langle x, y\rangle|$ for every $y \in H$.

Solution. Let $x_0 \in H_0$. Then choose $N \in \mathbb{N}$ and $c_1, \ldots, c_N \in \mathbb{F}$ such that $x_0 = \sum_{n=1}^{N} c_n\, u_n$. Since $\{u_1, \ldots, u_N\}$ is orthonormal, we have that $\|x_0\| = \big(\sum_{n=1}^{N} |c_n|^2\big)^{1/2}$.

Hence, using the hint and the Cauchy-Schwarz inequality in $\mathbb{F}^N$, we get

$$\|T_0(x_0)\| = \sup_{x\in H, \|x\|\le 1} |\langle x, T_0(x_0)\rangle| = \sup_{x\in H, \|x\|\le 1} \Big|\sum_{n=1}^{N} \langle x, v_n\rangle\, \overline{c_n}\Big|$$

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$$\leq \sup_{x\in H, \|x\|\leq 1} \Big(\big(\sum_{n=1}^{N} \big|\langle x, v_n\rangle\big|^2\big)^{1/2} \big(\sum_{n=1}^{N} |c_n|^2\big)^{1/2} \Big)$$

$$\leq \Big(\sup_{x\in H, \|x\|\leq 1} \sum_{n=1}^{\infty} \big|\langle x, v_n\rangle\big|^2 \Big)^{1/2} \|x_0\|$$

Now, since $\{v_n\}_{n\in\mathbb{N}}$ is a Bessel sequence, there exists some $M > 0$ such that $\big(\sum_{n=1}^{\infty} \big|\langle x, v_n\rangle\big|^2\big)^{1/2} \leq M^{1/2} \|x\|$ for all $x \in H$. Thus we have

$$\Big(\sup_{x\in H, \|x\|\leq 1} \sum_{n=1}^{\infty} \big|\langle x, v_n\rangle\big|^2 \Big)^{1/2} \leq M^{1/2}.$$

Combining this inequality with the one above, we get that

$$\|T_0(x_0)\| \leq M^{1/2} \|x_0\|.$$

This shows that T_0 is bounded, with $\|T_0\| \leq M^{1/2}$.

3c (weight 10 points)

Use 3b to show that T_0 may be extended to an operator $T \in \mathcal{B}(H)$ satisfying $T(u_n) = v_n$ for all $n \in \mathbb{N}$. Then deduce that for every $x \in H$, we have

$$T(x) = \sum_{n=1}^{\infty} \langle x, u_n\rangle \, v_n \, .$$

Solution. Since $\{u_n\}_{n\in\mathbb{N}}$ is an orthonormal basis for H, H_0 is dense in H. Hence, by the principle of extension by density and continuity, T_0 has a (unique) extension to an operator $T \in \mathcal{B}(H)$. For each $n \in \mathbb{N}$ we then have $T(u_n) = T_0(u_n) = v_n$, as desired. Moreover, by continuity and linearity of T, we get that

$$T(x) = T\Big(\sum_{n=1}^{\infty} \langle x, u_n\rangle \, u_n\Big) = \sum_{n=1}^{\infty} \langle x, u_n\rangle \, T(u_n) = \sum_{n=1}^{\infty} \langle x, u_n\rangle \, v_n$$

for each $x \in H$.

Problem 4 (weight 20 points)

Let H be a Hilbert space (over $\mathbb{R}$ or $\mathbb{C}$) and let $T \in \mathcal{B}(H)$ be self-adjoint.

4a (weight 10 points)

Assume that T has infinitely many eigenvalues. Show that T does not have finite-rank.

Solution. We can pick a sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ of distinct nonzero eigenvalues of T. Letting u_n be a unit eigenvector of T associated to λ_n for each n, we

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then have that $\{u_n\}_{n\in\mathbb{N}}$ is orthonormal (since the eigenspaces associated to different eigenvalues of T are orthogonal to each other). Further, we have that $u_n = (\lambda_n)^{-1}T(u_n) \in T(H)$ for each $n \in \mathbb{N}$. Thus, $T(H)$ is infinite-dimensional, i.e., T does not have finite-rank.

4b (weight 10 points)

Assume T is compact and has finitely many eigenvalues. Show that T has finite-rank.

Solution. If $T = 0$, the conclusion is obviously true. So we may assume that $T \neq 0$. We may use the spectral theorem for T. Letting L denote the set of nonzero eigenvalues of T, we then know that L is nonempty, and the assumption implies that L must be finite. For each $\lambda \in L$, let $\mathcal{E}_\lambda$ be an o.n.b. for the associated eigenspace $E_\lambda = \ker(T - \lambda I)$, which is finite-dimensional. Then the set $\mathcal{E}' = \cup_{\lambda\in L}\mathcal{E}_\lambda$ consists of finitely many vectors, and we know that it gives an o.n.b. for $\overline{T(H)}$. Since $T(H)$ is a subspace of $\overline{T(H)}$, this implies that $T(H)$ is finite-dimensional, i.e., T has finite-rank.