Exercises – Jan. 27, 2023

[L], 7.1.9 Let X be an uncountable set (e.g. $X = \mathbb{R}$)

Define $\mathcal{A} = \{ A \subseteq X \mid A$ or A^c is countable$\}$.

- $\mathcal{A}$ is a σ-algebra:

(i) $\emptyset \in \mathcal{A}$ since $\emptyset$ is countable (by def.)

(ii) Let $A \in \mathcal{A}$. Then $A^c \in \mathcal{A}$:

Two possibilities:
- A is countable. Then $(A^c)^c = A$ is countable, which means that $A^c \in \mathcal{A}$.
- A^c is countable. But this means again that $A^c \in \mathcal{A}$.

(iii) Let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathcal{A}$. Have to show that

$$\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{A}.$$

Assume first that A_n is countable for every $n \in \mathbb{N}$.

Then $\bigcup_{n\in\mathbb{N}} A_n$ is countable (cf. Spaces), so $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{A}$.

Next, assume that at least one of the A_n's, say A_{n_0}, is not countable. Since $A_{n_0} \in \mathcal{A}$, we must have that $(A_{n_0})^c$ is countable. But then we get

$$\Big(\bigcup_{n\in\mathbb{N}} A_n\Big)^c \underset{\text{De Morgan}}{=} \bigcap_{n\in\mathbb{N}} (A_n)^c \subseteq \underbrace{(A_{n_0})^c}_{\text{is countable}}$$

so $\Big(\bigcup_{n\in\mathbb{N}} A_n\Big)^c$ is countable, and it follows that $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{A}$.

7.1.9 (contin.) Define now $\mu : \mathcal{A} \to \mathbb{R}_+$ by

$$\mu(A) = \begin{cases} 0 & \text{if } A \text{ is countable} \\ 1 & \text{if } A^c \text{ is countable} \end{cases}, \quad A \in \mathcal{A}.$$

Then μ is a measure on $(X, \mathcal{A})$:

- $\mu(\emptyset) = 0$ since $\emptyset$ is countable.
- Let $\{A_n\}_{n \in \mathbb{N}}$ be a seq. of disjoint subsets in $\mathcal{A}$.

  Assume first that all A_n's are countable. Then $\bigcup_{n \in \mathbb{N}} A_n$ is countable, so we get

  $$\mu\Big(\bigcup_{n \in \mathbb{N}} A_n\Big) = 0 = \sum_{n=1}^{\infty} \mu(A_n)$$

  these are all zero

  Next, assume that A_{n_0} is not countable for at least one $n_0 \in \mathbb{N}$.

  We saw above that $\big(\bigcup_{n \in \mathbb{N}} A_n\big)^c$ is then countable, so

  we get that $\boxed{\mu\Big(\bigcup_{n \in \mathbb{N}} A_n\Big) = 1}$.

  On the other hand, consider $n \neq n_0$. Then

  $A_n \subseteq \underbrace{(A_{n_0})^c}_{\text{is countable}}$ (since A_n and A_{n_0} are disjoint),

  so A_n is countable, hence $\mu(A_n) = 0$. We get that $\boxed{\sum_{n \in \mathbb{N}} \mu(A_n) = \mu(A_{n_0}) = 1}$

  since $(A_{n_0})^c$ is countable

  Thus $\mu\Big(\bigcup_{n \in \mathbb{N}} A_n\Big) = \sum_{n=1}^{\infty} \mu(A_n)$

  also in this case, as desired.

7.1.17 $X \neq \emptyset$, $\mathcal{A} \subseteq \mathcal{P}(X)$.

$\mathcal{A}$ is called an algebra (on X) when

(i) $\emptyset \in \mathcal{A}$, (ii) $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$, (iii) $A, B \in \mathcal{A} \Rightarrow A \cup B \in \mathcal{A}$.

Then we have

a) $A_1, \dots, A_n \in \mathcal{A} \Rightarrow \bigcup_{k=1}^n A_k \in \mathcal{A}$ [Follows easily by induction on n, using (iii)]

b) — " — $\Rightarrow \bigcap_{k=1}^n A_k \in \mathcal{A}$ [Use that $\left(\bigcap_{k=1}^n A_k\right)^c = \bigcup_{k=1}^n (A_k)^c$]

c) $X = \mathbb{N}$, $\mathcal{A} = \{A \subseteq \mathbb{N} : A \text{ or } A^c \text{ is finite}\}$

Then proceeding much in the same way as in the prev. exercise one checks that $\mathcal{A}$ is an algebra.

But $\mathcal{A}$ is not a σ-algebra:

Set $A_n := \{2n\}$, $n \in \mathbb{N}$; so $A_n \in \mathcal{A}$ for each n.

Then $\bigcup_{n \in \mathbb{N}} A_n$ = all even numbers $\to$ is infinite

$\left(\bigcup_{n \in \mathbb{N}} A_n\right)^c$ = all odd numbers $\to$ also infinite

Hence $\bigcup_{n \in \mathbb{N}} A_n \notin \mathcal{A}$.

d) Assume $\mathcal{A}$ is an algebra which is closed under countable union of disjoint sets. Then $\mathcal{A}$ is a σ-algebra:

Let $\{A_n\}$ be a seq. in $\mathcal{A}$.

Then set $B_1 = A_1$, $B_n = \underbrace{\underbrace{A_n}_{\in \mathcal{A}} \setminus \underbrace{\left(\bigcup_{k=1}^{n-1} A_k\right)}_{\in \mathcal{A}}}_{\in \mathcal{A}}$, $n \geq 2$.

As in the lemma from yesterday, we have that the B_n's are disjoint and $\bigcup_{n \in \mathbb{N}} A_n = \bigcup_{n \in \mathbb{N}} B_n$.

From the assumption, we get that $\bigcup_{n \in \mathbb{N}} B_n \in \mathcal{A}$.

So $\bigcup_{n \in \mathbb{N}} A_n \in \mathcal{A}$.

Extra-exercise 1 $\quad X \neq \emptyset$, $A \subseteq X$, $g: X \to \overline{\mathbb{R}}_+ = [0,\infty]$

$$\sum_{x\in A} g(x) = \sup\left\{ \sum_{x\in F} g(x) : F \subseteq A, F \text{ finite} \right\} \quad \text{if } A \neq \emptyset$$

($A = \emptyset \rightsquigarrow$ the sum is zero)

a) Assume A is countably infinite, so $A = \{a_1, a_2, \dots\}$ without repetitions

Then we have that

$$\boxed{\sum_{x\in A} g(x) = \sum_{i=1}^{\infty} g(a_i)} \quad \left(\overset{\text{def}}{:=} \lim_{N\to\infty} \sum_{i=1}^{N} g(a_i) \right)$$

Let $N \in \mathbb{N}$. Then $\sum_{i=1}^{N} g(a_i) = \sum_{x\in\{a_1,\dots,a_N\}} g(x) \leq \sum_{x\in A} g(x)$

Taking the limit, we get that $\sum_{i=1}^{\infty} g(a_i) \leq \sum_{x\in A} g(x)$.

Conversely, let $F \subseteq A$ be finite. So $F = \{a_{j_1}, a_{j_2}, \dots, a_{j_k}\}$ for some $j_1, j_2, \dots, j_k \in \mathbb{N}$.

Set $N = \max\{j_1, \dots, j_k\}$.

Then $\sum_{x\in F} g(x) = \sum_{i=1}^{k} g(a_{j_i}) \leq \sum_{j=1}^{N} g(a_j) \leq \sum_{j=1}^{\infty} g(a_j)$

So $\underbrace{\sup\left\{ \sum_{x\in F} g(x) \mid F \subseteq A, F \text{ finite} \right\}}_{\sum_{x\in A} g(x)} \leq \sum_{j=1}^{\infty} g(a_j)$

b) Assume $\sum_{x\in A} g(x) < \infty$. Set $A_g := \{x\in A : g(x) \neq 0\}$.

Then A_g is countable and $\sum_{x\in A} g(x) = \sum_{x\in A_g} g(x)$:

Let $n \in \mathbb{N}$ and set $A_n := \{x \in A \mid g(x) \geq \frac{1}{n}\}$.

Then A_n is finite. Indeed, assume (for contradiction) that A_n is infinite. We can then pick $\{x_k\}_{k\in\mathbb{N}}$ in A_n where $x_k \neq x_{k+1}$ for all k. Then

$\sum_{k=1}^{m} g(x_k) \geq m\cdot\frac{1}{n}$. So we get $\sum_{x\in A_n} g(x) \geq \underbrace{m\cdot\frac{1}{n}}_{\infty \text{ when } m\to\infty}$

i.e. $\sum_{x\in A_n} g(x) = \infty$. But

$\sum_{x\in A_n} g(x) \leq \sum_{x\in A} g(x) < \infty$ ↑ by assumption

So we get a contradiction.

We therefore get that $A_g = \bigcup_{n\in\mathbb{N}} A_n$ must be countable.

Then $\sum_{x\in A} g(x) = \sum_{x\in A_g} g(x) + \sum_{x\in A\setminus A_g} g(x) \rightsquigarrow g(x) = 0 \;\forall x \notin A_g$

$= \sum_{x\in A_g} g(x)$

Ex. Exercise 2 $X \neq \emptyset$ $g: X \to [0,\infty]$

$$\mu_g(A) := \sum_{x \in A} g(x), \quad A \in \mathcal{P}(X).$$

Then μ_g is a measure on $(X, \mathcal{P}(X))$:

Let $\{A_n\}$ be a sequence of disjoint subsets of X. We have to show that $\mu_g\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \sum_{n=1}^{\infty} \mu_g(A_n)$, that is,

$$(*) \quad \boxed{\sum_{x\in A} g(x) = \sum_{n=1}^{\infty}\left(\sum_{x\in A_n} g(x)\right) \quad \text{where } A := \bigcup_{n\in\mathbb{N}} A_n}$$

- If $g(x) = \infty$ for some $x \in A$, we get ∞ on both sides.
- Assume $g(x) < \infty$ for all $x \in A$.

  Let $F \subseteq A$ be finite, and set $F_n := F \cap A_n$, $n \in \mathbb{N}$.

  Set $I = \{n \in \mathbb{N} : F_n \neq \emptyset\}$.

  Since $F = \bigcup_{i\in I} F_i$, I is finite.

$$\text{So } \sum_{x\in F} g(x) = \sum_{i\in I}\left(\sum_{x\in F_i} g(x)\right) \le \sum_{i\in I}\left(\sum_{x\in A_i} g(x)\right)$$

$$\le \sum_{i=1}^{\infty}\left(\sum_{x\in A_i} g(x)\right)$$

Taking the sup. over all F's as above, we get

$$\sum_{x\in A} g(x) \le \sum_{i=1}^{\infty}\left(\sum_{x\in A_i} g(x)\right)$$

Note that if $\sum_{x\in A} g(x) = \infty$, then this gives that

$$\sum_{i=1}^{\infty}\left(\sum_{x\in A_i} g(x)\right) = \infty,$$ so we have equality.

So assume now that $M := \sum_{x\in A} g(x) < \infty$.

It remains to show that $\sum_{i=1}^{\infty}\left(\sum_{x\in A_i} g(x)\right) \le M$

The rest of the proof and the solution of Extra-ex. 3 will be put on the homepage!