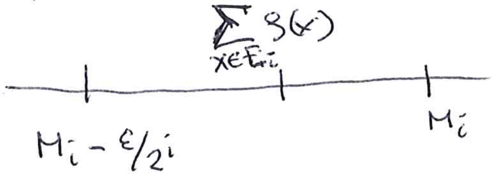
Here is the end of the proof of a) in Extra-Exercise 2

We assume that $\underbrace{\sum_{x \in A} g(x)}_{=: M} < \infty$, and have to show

that $$\boxed{\sum_{i=1}^{\infty} \Big(\sum_{x \in A_i} g(x) \Big) \leq M} \quad (**)$$ (we recall that $A = \bigcup_{i \in \mathbb{N}} A_i$, where the A_i's are disjoint)

Let $i \in \mathbb{N}$. We note that if F is a finite subset of A_i, then F is a finite subset of A, and it readily follows that $\sum_{x \in A_i} g(x) \leq \sum_{x \in A} g(x) = M < \infty$. Thus, for each $i \in \mathbb{N}$, we have $M_i := \sum_{x \in A_i} g(x) < \infty$.



Let $\varepsilon > 0$.

For each $i \in \mathbb{N}$, we can then pick $E_i \subseteq A_i$, E_i finite, such that $M_i - \varepsilon/2^i < \sum_{x \in E_i} g(x)$.

Let $n \in \mathbb{N}$. Then we have

$$\sum_{i=1}^{n} M_i < \sum_{i=1}^{n} \Big(\sum_{x \in E_i} g(x) + \varepsilon/2^i \Big) = \sum_{x \in \bigcup_{i=1}^{n} E_i} g(x) + \varepsilon \cdot \sum_{i=1}^{n} 1/2^i$$

$$\leq \sum_{x \in A} g(x) + \varepsilon \cdot \underbrace{\sum_{i=1}^{\infty} \left(\tfrac{1}{2}\right)^i}_{= 1} = M + \varepsilon.$$

letting $n \to \infty$, we get $\sum_{i=1}^{\infty} M_i < M + \varepsilon$.

Since this holds for every $\varepsilon > 0$, we get that

$\sum_{i=1}^{\infty} M_i \leq M$, i.e. $(**)$ holds, as desired.

Extra-exercise 2 b) $X = \mathbb{R}$, $\rho(x) = \begin{cases} 1/x^2 & , x \neq 0 \\ \infty & , x = 0 \end{cases}$,
$\mu_\rho(A) = \sum_{x \in A} \rho(x)$.

- $A = \{x_0\}$ gives $\mu_\rho(A) = \rho(x_0) = \begin{cases} 1/x_0^2 & \text{if } x_0 \neq 0 \\ \infty & \text{if } x_0 = 0. \end{cases}$

- $A = \mathbb{N}$ gives $\mu_\rho(A) = \sum_{i \in \mathbb{N}} \rho(i) = \sum_{i=1}^{\infty} \frac{1}{i^2} = \pi^2/6$.

- $A = \mathbb{Z}$ gives $\mu_\rho(A) = \infty$ (since $0 \in A$ and $\rho(0) = \infty$)

- $A = (0,1]$ gives $\mu_\rho(A) = \infty$:

Indeed, for each $n \in \mathbb{N}$, set $F_n = \{1, 1/\sqrt{2}, \dots, 1/\sqrt{n}\} \subseteq (0,1]$.

Then $\sum_{x \in F_n} \rho(x) = \sum_{j=1}^{n} \frac{1}{(1/\sqrt{j})^2} = \sum_{j=1}^{n} j = \frac{n(n+1)}{2}$.

So $\frac{n(n+1)}{2} \leq \sum_{x \in A} \rho(x) = \mu_\rho(A)$ for every $n \in \mathbb{N}$.

This implies that $\mu_\rho(A) = \infty$.

Extra-exercise 3

Consider $\{t_{(n,k)}\}_{(n,k)\in\mathbb{N}\times\mathbb{N}} \subseteq \overline{\mathbb{R}}_+$ and

$$S := \sum_{(n,k)\in\mathbb{N}\times\mathbb{N}} t_{(n,k)} .$$

a) Let $\sigma : \mathbb{N} \to \mathbb{N}\times\mathbb{N}$ be a bijection and $s_m := t_{\sigma(m)}$, $m\in\mathbb{N}$.

We want to show that

$$\boxed{S \overset{(1)}{=} \sum_{m=1}^{\infty} s_m \overset{(2)}{=} \sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty} t_{(n,k)}\right) \overset{(3)}{=} \sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty} t_{(n,k)}\right)}$$

Proof of (1): • Let $M\in\mathbb{N}$. Set $F_M = \{\sigma(1), \sigma(2), \dots, \sigma(M)\}$. Then we have $\sum_{m=1}^{M} s_m = \sum_{(n,k)\in F_M} t_{(n,k)} \le S$.

So $\sum_{m=1}^{\infty} s_m = \lim_{M\to\infty} \sum_{m=1}^{M} s_m \le S$.

• Let $F \subseteq \mathbb{N}\times\mathbb{N}$, F finite. Set $M_F := \max(\sigma^{-1}(F)) \in \mathbb{N}$.

Then $\sum_{(n,k)\in F} t_{(n,k)} = \sum_{m\in\sigma^{-1}(F)} s_m \le \sum_{m=1}^{M_F} s_m \le \sum_{m=1}^{\infty} s_m$.

• This shows that $\sum_{m=1}^{\infty} s_m = S$.

Proof of 2 :

- Let $N, K \in \mathbb{N}$. Then $\sum_{n=1}^{N}\left(\sum_{k=1}^{K} t_{(n,k)}\right) = \sum_{\substack{(n,k)\in \\ \{1,\dots,N\}\times\{1,\dots,K\}}} t_{(n,k)} \leq S$.

  Letting $K \to \infty$, we get $\sum_{n=1}^{N}\left(\sum_{k=1}^{\infty} t_{(n,k)}\right) \leq S$.

  Letting $N \to \infty$, we get $\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty} t_{(n,k)}\right) \leq S$.

- Let $F \subseteq \mathbb{N}\times\mathbb{N}$, F finite. Choose $N, K \in \mathbb{N}$ s.t. $F \subseteq \{1,\dots,N\}\times\{1,\dots,K\}$.

  Then $\sum_{(n,k)\in F} t_{(n,k)} \leq \sum_{n=1}^{N}\left(\sum_{k=1}^{K} t_{(n,k)}\right) \leq \sum_{n=1}^{N}\left(\sum_{k=1}^{\infty} t_{(n,k)}\right) \leq \sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty} t_{(n,k)}\right)$.

  So $S := \sup\left\{\sum_{(n,k)\in F} t_{(n,k)} : F \subseteq \mathbb{N}\times\mathbb{N}, F \text{ finite}\right\} \leq \sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty} t_{(n,k)}\right)$

- This shows that $S = \sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty} t_{(n,k)}\right)$.

The proof of 3 is similar.

b) Assume $\{A_{(n,k)}\}_{(n,k)\in\mathbb{N}\times\mathbb{N}} \subseteq \mathcal{P}(X)$, $\mu : \mathcal{P}(X) \to [0,\infty]$.

For $\sigma : \mathbb{N} \to \mathbb{N}\times\mathbb{N}$ bijection, set $B_m := A_{\sigma(m)}$, $m \in \mathbb{N}$.

Setting $t_{(n,k)} := \mu(A_{(n,k)})$ for each (n,k), we get $\mu(B_m) = t_{\sigma(m)}$ for each $m \in \mathbb{N}$. Using a), we get

$$\underline{\sum_{(n,k)\in\mathbb{N}\times\mathbb{N}} \mu(A_{(n,k)})} = \sum_{(n,k)\in\mathbb{N}\times\mathbb{N}} t_{(n,k)} = \sum_{m=1}^{\infty} t_{\sigma(m)} = \underline{\sum_{m=1}^{\infty} \mu(B_m)}$$

$$= \sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty} t_{(n,k)}\right) = \underline{\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty} \mu(A_{(n,k)})\right)}$$

$$= \sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty} t_{(n,k)}\right) = \underline{\sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty} \mu(A_{(n,k)})\right)}$$