MAT3400/4400 - Spring 19 - Exercises for Friday, Mars 10

Extra-exercise 14

Let $\mathcal{J}$ denote the family of subsets of $\mathbb{R}$ given by

$$\mathcal{J} = \{\emptyset\} \cup \big\{(a,b] : a,b \in \mathbb{R},\ a < b\big\} \cup \big\{(-\infty, b] : b \in \mathbb{R}\big\} \cup \big\{(a,\infty) : a \in \mathbb{R}\big\},$$

(already met in Extra-exercise 4)

Check that $\mathcal{J}$ is a semialgebra.

Extra-exercise 15

Let $\mathcal{S}$ be a semialgebra of subsets of a nonempty set X and let λ be a premeasure on $\mathcal{S}$. Let $\mathcal{R}$ be the algebra consisting of all unions of finitely many disjoint sets in $\mathcal{S}$ and let ρ be the premeasure on $\mathcal{R}$ given by $\rho(A) = \sum_{j=1}^{n} \lambda(S_j)$ whenever $A = S_1 \cup \cdots \cup S_n$ for some disjoint sets $S_1, \ldots, S_n$ in $\mathcal{S}$.

Let ν^* denote the outer measure on $\mathcal{P}(X)$ associated with $\mathcal{S}$ and λ, and let μ^* denote the outer measure on $\mathcal{P}(X)$ associated with $\mathcal{R}$ and ρ.

Check that $\nu^* = \mu^*$. (This is the content of the remark on p. 302 in [L]). This means that

$$\mu^*(A) = \inf\Big\{ \sum_{j=1}^{\infty} \lambda(S_j) : \{S_j\}_{j\in\mathbb{N}} \text{ is a } \mathcal{S}\text{-covering of } A\Big\}$$

for every $A \subseteq X$.

- From Lindstrøm's book, Section 8.3: 3

- From Lindstrøm's book, Section 8.4: 2, 3, 5

Note. In exercises 2 and 5 above, measurable means Lebesgue measurable.

Extra-exercise 16

Let $E \subseteq \mathbb{R}$ be Borel measurable, and let $a, r \in \mathbb{R}$, $r \neq 0$.

Show that $E + a$ and rE are also Borel measurable.

Hint. Consider the collections of subsets of $\mathbb{R}$ given by

$\mathcal{B} + a := \{E + a \mid E \text{ is a Borel subset of } \mathbb{R}\}$ and $r\mathcal{B} := \{rE \mid E \text{ is a Borel subset of } \mathbb{R}\}$.