MAT3400/4400 - Spring 2023 - Exercises for Friday, Mars 17

Extra exercise 17

Let $\mathcal{L}$ denote the σ-algebra of all Lebesgue measurable subsets of $\mathbb{R}$ and let μ denote the Lebesgue measure on $\mathcal{L}$.

Let S denote either $\overline{\mathbb{R}}$ or $\mathbb{C}$, let $f : \mathbb{R} \to S$ and $a \in \mathbb{R}$. Define $f_a : \mathbb{R} \to S$ by

$$f_a(x) := f(x - a) \quad \text{for all } x \in \mathbb{R}.$$

a) Show that f is Lebesgue measurable if and only if f_a is Lebesgue measurable.

b) Assume that $g : \mathbb{R} \to \overline{\mathbb{R}}_+$ is Lebesgue measurable, so $g_a : \mathbb{R} \to \overline{\mathbb{R}}_+$ is also Lebesgue measurable (by a)). Show that

$$\int_{E+a} g_a \, d\mu = \int_E g \, d\mu$$

for every $E \in \mathcal{L}$. (We recall that $E + a := \{e + a \mid e \in E\} \in \mathcal{L}$.)

c) Consider again $f : \mathbb{R} \to S$ and $E \in \mathcal{L}$. Show that f is integrable over E if and only if f_a is integrable over $E + a$, in which case we have

$$\int_{E+a} f_a \, d\mu = \int_E f \, d\mu \, .$$

Extra exercise 18

Let C denote the Cantor set. We have seen that C is a closed subset of $[0, 1]$ (hence is compact and Lebesgue measurable) which has Lebesgue measure zero. Our description of C was geometric, but C can also be defined using ternary expansions of numbers in $[0, 1]$, namely as

$$C = \Big\{ \sum_{n=1}^{\infty} \frac{a_n}{3^n} : a_n \in \{0, 2\} \text{ for each } n \in \mathbb{N} \Big\}.$$

(cf. Brevig's notes, Sect. 1.3, in particular the proof of Theorem 1.3.3). In fact, setting

$$S := \Big\{ \{a_n\}_{n\in\mathbb{N}} : a_n \in \{0, 2\} \text{ for each } n \in \mathbb{N} \Big\},$$

the map $g : S \to C$ given by

$$g\Big(\{a_n\}_{n\in\mathbb{N}}\Big) = \sum_{n=1}^{\infty} \frac{a_n}{3^n}$$

is a bijection. Let now $h : S \to [0, 1]$ be given by

$$h\Big(\{a_n\}_{n\in\mathbb{N}}\Big) = \sum_{n=1}^{\infty} \frac{a_n/2}{2^n}.$$

Show that h is surjective, but not injective. Deduce that $f := h \circ g^{-1}$ is a surjective map from C onto $[0, 1]$, and that C is therefore uncountable.

Note. It can be shown that there exists a bijection between C and $[0, 1]$, i.e., that these sets have the same cardinality, but this requires the so-called Schröder-Bernstein theorem from set theory.

- Exercise from Lindstrøm's book: 8.5.2

Extra exercise 19

Let $\mathcal{L}$ denote the σ-algebra of all Lebesgue measurable subsets of $\mathbb{R}$ and $\mathcal{B}$ denote the σ-algebra of all Borel subsets of $\mathbb{R}$. Let μ denote the Lebesgue measure on $\mathcal{B}$.

It can be shown that $\mathcal{B} \neq \mathcal{L}$. However, Lebesgue measurable sets can related to Borel sets as follows. Set

$$\mathcal{N} := \{N \subseteq \mathbb{R} : \text{there exists } C \in \mathcal{B} \text{ such that } N \subseteq C \text{ and } \mu(C) = 0\Big\},$$

and let $A \subseteq \mathbb{R}$. Show that $A \in \mathcal{L}$ and only if there exists $B \in \mathcal{B}$ and $N \in \mathcal{N}$ such that

$$A = B \cup N\,.$$

Extra exercise 20

Let $A \subseteq \mathbb{R}$ be Lebesgue measurable and let μ denote the Lebesgue measure on the Lebesgue measurable subsets of $\mathbb{R}$. Show that

$$\begin{aligned}\mu(A) &= \inf\big\{\mu(G) \mid G \text{ is open in } \mathbb{R} \text{ and } A \subseteq G\big\} \\ &= \sup\big\{\mu(K) \mid K \text{ is compact in } \mathbb{R} \text{ and } K \subseteq A\big\}.\end{aligned}$$

- Exercises from Brevig's notes: 2.1 and 2.2