MAT3400/4400 - Spring 2023 - Exercises for Friday, Mars 24

Extra exercise 21

Consider $X = [-1, 1]$ with the Lebesgue measure μ on the Lebesgue measurable subsets of X, and the function $f = \mathbf{1}_{[0,1]}$ on X.

Show that there exists no continuous function $g : X \to \mathbb{R}$ such that $f = g \quad \mu$-a.e.

Extra exercise 22

Let $f : [a, b] \to \mathbb{R}$ be monotone (increasing or decreasing) and bounded.

Show that the set $D_f := \{x \in [a, b] : f \text{ is not continuous at } x\}$ is countable. Deduce from Lebesgue's criterion that f is Riemann-integrable.

Extra exercise 23

Set $\mathcal{L}^1(\mathbb{R}, \mathbb{R}) := \{f : \mathbb{R} \to \mathbb{R} : f \text{ is Lebesgue integrable}\}$. For simplicity we just write $\mathcal{L}^1(\mathbb{R})$ for this space here (the point being that we could also have considered $\mathcal{L}^1(\mathbb{R}, \mathbb{C})$, as we will do later).

a) Show that $\mathcal{L}^1(\mathbb{R})$ is a subspace of the vector space of all real-valued functions on $\mathbb{R}$ (with pointwise operations).

b) Assume g_1, g_2 are real-valued functions on $\mathbb{R}$ which are both compactly supported, and let $c_1, c_2 \in \mathbb{R}$. Check that $c_1\, g_1 + c_2\, g_2$ is compactly supported too.

Deduce that $C_c(\mathbb{R}) := \{g : \mathbb{R} \to \mathbb{R} : g \text{ is continuous and compactly supported}\}$ is a subspace of $\mathcal{L}^1(\mathbb{R})$.

- Exercises from Brevig's notes: 2.5, 2.6 and 2.7

Extra exercise 24

Show that Littlewood's third principle (i.e., Egorov's theorem) still holds if one instead assumes that the sequence $\{f_n\}$ converges to f pointwise on X μ-a.e.

Extra exercise 25

Consider the following statement:

Assume $(X, \mathcal{A}, \mu)$ is a finite measure space and $\{f_n\}$ and f are measurable real-valued functions on X such that $\{f_n\}$ converges to f pointwise on X. Then there exists some $A \in \mathcal{A}$ such that $\mu(A) = 0$ and $\{f_n\}$ converges uniformly on A^c.

Either prove this statement or find a counterexample.

- Exercises from ELA (= the notes on Elem. Linear Analysis): 1.2 and 1.3