MAT3400/4400 - Spring 2023 - Exercises for Friday, April 14

Extra Exercise 27

Consider the measure space $([0,1], \mathcal{A}, \mu)$ where $\mathcal{A}$ denotes the Lebesgue measurable subsets of $[0,1]$ and μ is the Lebesgue measure on $(X, \mathcal{A})$. Let $p \in [1,\infty)$ and $\{f_n\}_{n\in\mathbb{N}}$ be the sequence of functions on $[0,1]$ given by

$$f_1 = 1_{[0,1]},$$

$$f_2 = 1_{[0,1/2]}, \quad f_3 = 1_{[1/2,1]},$$

$$f_4 = 1_{[0,1/3]}, \quad f_5 = 1_{[1/3,2/3]}, \quad f_6 = 1_{[2/3,1]},$$

$$f_7 = 1_{[0,1/4]}, \quad f_8 = 1_{[1/4,1/2]}, \quad f_9 = 1_{[1/2,3/4]}, \quad f_{10} = 1_{[3/4,1]},$$

etc

Show that the sequence $\{f_n\}_{n\in\mathbb{N}}$ converges to the zero function on X w.r.t. $\|\cdot\|_p$, and that $\{f_n(x)\}_{n\in\mathbb{N}}$ is divergent for every $x \in [0,1]$.

Extra Exercise 28

Let X be a nonempty set, $\mathcal{A} := \mathcal{P}(X)$ (the σ-algebra consisting of all subsets of X), and μ denote the counting measure on $(X, \mathcal{A})$. Note that every complex-valued function on X is then measurable (w.r.t. $\mathcal{A}$).

a) Let $\rho : X \to \mathbb{R}_+$. Show that

$$\int_X \rho \, d\mu = \sum_{x\in X} \rho(x)$$

where the sum $\sum_{x\in X} \rho(x)$ is defined as in the first exercise set (for Jan. 27). Then deduce that the measure $\mu_\rho : \mathcal{P}(X) \to [0,\infty]$ defined by

$$\mu_\rho(A) = \sum_{x\in A} \rho(x)\,,$$

satisfies that $\mu_\rho(A) = \int_A \rho \, d\mu$ for all $A \in \mathcal{P}(X)$.

b) Let $p \in [1,\infty)$ and let $f : X \to \mathbb{C}$. Deduce from a) that $\|f\|_p = \big(\sum_{x\in X} |f(x)|^p\big)^{1/p}$, hence that

$$\mathcal{L}^p(X, \mathcal{A}, \mu) = \Big\{ f : X \to \mathbb{C} : \sum_{x\in X} |f(x)|^p < \infty \Big\}.$$

Verify also that $\|\cdot\|_p$ is a norm on $\mathcal{L}^p(X, \mathcal{A}, \mu)$, hence that $L^p(X, \mathcal{A}, \mu) = \mathcal{L}^p(X, \mathcal{A}, \mu)$. As is common, we set $\ell^p(X) := \mathcal{L}^p(X, \mathcal{A}, \mu)$.

c) Let $1 \le p \le r < \infty$ and let $\ell^\infty(X)$ denote the space of all bounded complex-valued functions on X. Show that

$$\ell^p(X) \subseteq \ell^r(X) \subseteq \ell^\infty(X).$$

- Exercises from Chap. 2 of the Notes on ELA: 12, 13, 14, 15, 16