MAT3400/4400 - Spring 2023 - Exercises for Friday, Jan. 27

Exercises from Lindstrøm's book, Section 7.1: 9, 17

Some extra exercises

We recall that $\overline{\mathbb{R}}_+ := [0, \infty]$ and that addition in $\mathbb{R}_+ = [0, \infty)$ is extended to $\overline{\mathbb{R}}_+$ by setting

$$x + \infty = \infty + x := \infty \quad \text{for all } x \in \overline{\mathbb{R}}_+ .$$

We also extend the usual order $\leq$ on $\mathbb{R}_+ = [0, \infty)$ to $\overline{\mathbb{R}}_+$ by setting

$$x \leq \infty \quad \text{for all } x \in \mathbb{R}_+ .$$

Note that any subset S of $\overline{\mathbb{R}}_+$ has a least upper bound in $\overline{\mathbb{R}}_+$, which is denoted by $\sup S$. We have $\sup S = \infty$ if S contains ∞ or if S is an unbounded subset of $\mathbb{R}_+$, while $\sup S$ is the usual least upper bound of S in $\mathbb{R}_+$ if $S \subseteq \mathbb{R}_+$ is bounded.

Let now X be a nonempty set, $A \subseteq X$ and $\rho : X \to [0, \infty]$. We recall how the sum $\sum_{x \in A} \rho(x)$ is defined.

• If A is empty, we sum over nothing, so we set $\sum_{x \in A} \rho(x) := 0$.

• Assume A is nonempty and finite. Letting $a_1, a_2, \ldots, a_n$ be any listing of the elements of A (without any repetition), we set

$$\sum_{x \in A} \rho(x) := \sum_{j=1}^{n} \rho(a_j) .$$

• If A is infinite, we set

$$\sum_{x \in A} \rho(x) := \sup \Big\{ \sum_{x \in F} \rho(x) \,|\, F \subseteq A \,,\, F \text{ finite} \Big\}.$$

Note: One often meets sums of the form $\sum_{j \in J} t_j$, where $\{t_j\}_{j \in J}$ is a family of elements in $\overline{\mathbb{R}}_+$ indexed by a nonempty set J. Such a sum is simply defined by

$$\sum_{j \in J} t_j := \sum_{j \in J} \rho(j)$$

where $\rho : J \to \overline{\mathbb{R}}_+$ is given by $\rho(j) := t_j$ for each $j \in J$.

In the next two exercises, X and ρ are as above.

Exercise 1

Let $A \subseteq X$.

a) Assume that A is countably infinite and let $a_1, a_2, a_3, \ldots$ be a listing of the elements of A (without any repetition). Check that

$$\sum_{x \in A} \rho(x) = \sum_{i=1}^{\infty} \rho(a_i)$$

(where $\sum_{i=1}^{\infty} \rho(a_i)$ has its obvious meaning).

b) Assume $\sum_{x \in A} \rho(x) < \infty$ and set $A_\rho := \{x \in A : \rho(x) \neq 0\}$. Show that A_ρ is countable and

$$\sum_{x \in A} \rho(x) = \sum_{x \in A_\rho} \rho(x).$$

Then use a) to deduce that if A_ρ is countably infinite and $a_1, a_2, a_3, \ldots$ is a listing of the elements of A_ρ (without any repetition), then

$$\sum_{x \in A} \rho(x) = \sum_{i=1}^{\infty} \rho(a_i)\,.$$

Exercise 2

Recall that $\mathcal{P}(X)$ denote the σ-algebra on X which consists of all subsets of X.

Define $\mu_\rho : \mathcal{P}(X) \to [0, \infty]$ by

$$\mu_\rho(A) = \sum_{x \in A} \rho(x)\,, \quad A \in \mathcal{P}(X)\,.$$

a) Show that μ_ρ is measure on $\mathcal{P}(X)$.

b) Consider $X = \mathbb{R}$ and $\rho : \mathbb{R} \to \overline{\mathbb{R}}_+$ given by

$$\rho(x) = \begin{cases} 1/x^2 & \text{if } x \neq 0, \\ \infty & \text{if } x = 0. \end{cases}$$

Let $A \subseteq \mathbb{R}$ and $x_0 \in \mathbb{R}$. Find $\mu_\rho(A)$ in the following cases:

$A = \{x_0\}$; $A = \mathbb{N}$; $A = \mathbb{Z}$; $A = (0, 1]$.

Remark. Exercises 1, 2, 3, 4 in Section 7.1 of Lindstrøm's book are special cases of Exercise 2 a). If you have trouble with solving it, you should try to solve these four exercises first as a training.

Exercise 3

Let $\{t_{(n,k)}\}_{(n,k)\in\mathbb{N}\times\mathbb{N}}$ be a family of numbers in $\overline{\mathbb{R}}_+$ indexed by $\mathbb{N}\times\mathbb{N}$. Set

$$S := \sum_{(n,k)\in\mathbb{N}\times\mathbb{N}} t_{(n,k)} \in [0,\infty]\,.$$

a) Since $\mathbb{N}\times\mathbb{N}$ is countable, there exists a bijective map $\sigma:\mathbb{N}\to\mathbb{N}\times\mathbb{N}$. Set $s_m = t_{\sigma(m)}$ for each $m\in\mathbb{N}$. Show that

$$S = \sum_{m=1}^{\infty} s_m = \sum_{n=1}^{\infty}\Big(\sum_{k=1}^{\infty} t_{(n,k)}\Big) = \sum_{k=1}^{\infty}\Big(\sum_{n=1}^{\infty} t_{(n,k)}\Big)\,.$$

b) Let X be a set and $\mu:\mathcal{P}(X)\to[0,\infty]$ be a map.

For each $(n,k)\in\mathbb{N}\times\mathbb{N}$, let $A_{(n,k)}$ be a subset of X. For a bijection σ as in a), set $B_m = A_{\sigma(m)}$ for each $m\in\mathbb{N}$. Deduce from a) that

$$\sum_{(n,k)\in\mathbb{N}\times\mathbb{N}} \mu\big(A_{(n,k)}\big) = \sum_{m=1}^{\infty} \mu(B_m) = \sum_{n=1}^{\infty}\Big(\sum_{k=1}^{\infty} \mu\big(A_{(n,k)}\big)\Big) = \sum_{k=1}^{\infty}\Big(\sum_{n=1}^{\infty} \mu\big(A_{(n,k)}\big)\Big)\,.$$